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# THE
# ELASTIC STRENGTH OF GUNS

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THE

# ELASTIC STRENGTH OF GUNS

PREPARED AND ARRANGED

AS A TEXT BOOK

FOR THE USE OF CADETS AT THE U. S.
NAVAL ACADEMY.

[illegible]

*Department of Ordnance and Gunnery.*

1891

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Baltimore, Md.

# PREFACE.

The edition of the Elastic Strength of Guns published in 1885, compiled by Lieuts. J. F. Meigs and R. R. Ingersoll, having been exhausted, this second edition has been prepared for the use of the Naval Cadets, and therefore treats of such applications of the theory as is deemed necessary for the Course in Ordnance and Gunnery at this Academy.

Much of the first edition has been retained without change and examples for work in the section-room have been added. A chapter giving the elementary principles governing the construction of Wire Wound Guns has been introduced, in view of the development probable in that form of construction.

R. R. INGERSOLL,
*Lieut. U. S. Navy.*

U. S. NAVAL ACADEMY, MAY 1891.

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# CHAPTER I.

## FUNDAMENTAL LAWS OF THE ELASTICITY OF SOLIDS.

### § I.

HOOKE'S LAW. This law, called from its discoverer, Robert Hooke, is, that, within certain limits of *stress* and *strain*, the latter increases uniformly with the former.

The term *stress* is here used to denote the force or load applied to the material, and the term *strain* denotes the corresponding change in dimension due to the particular load applied.

It has been found, by experiment, that Hooke's law is true for the metals used in the construction of Guns to certain limits, and the limit of *strain* within which the law is true is called the *Elastic Limit*. The intensity of the force or load acting when the *strain* has reached the elastic limit is called the *Elastic Strength;* the latter is usually given in pounds or tons per square inch and the former in inches per unit of dimension.

The *elastic limit* for metals used for guns is generally a very small quantity. If $l$ is the length in inches, before the load is applied, of a bar of any material, and if $x$ is the total change of length due to any given stress or load, the change or *strain* per unit length is

$$e = \frac{x}{l}$$

and, if $S$ is the stress per unit cross-section, per square inch, say, we have, within the *elastic strength* and *elastic limit*, by Hooke's law, that the ratio of the *stress* to the *strain* produced by that *stress* is a constant quantity.

If $E$ denotes this quantity, we have

$$\frac{S}{e} = E.$$

The constant $E$ is called the *Modulus of Elasticity* for the particular material used. As an example; a bar of steel, 10 inches in length before a load is applied, and having a natural cross-section of $\frac{1}{2}$ square inch, is subjected to a pull of 20,000 pounds, or 40,000 pounds per square inch; the bar having increased in length under stress by $\frac{1}{750}$ of an inch, it is required to find the Modulus of Elasticity for this steel.

Answer: $E = 30,000,000$ pounds (the stress in this case is assumed to be less than the elastic strength). Evidently the cross-section under stress will be a little less than the natural cross-section before the load was applied, but the stress is notwithstanding referred to the natural cross-section; obviously, this is a safe rule in practice.

The elongation is also tacitly assumed to have taken place uniformly throughout the length of the bar, which will not be strictly true in practice, because it has been found that long bars yield more at some points of their length than do short bars, and therefore will give a smaller value to the *elastic limit* than will the same metal in shorter bars.

The *Modulus of Elasticity* of Gun steel as used by the Naval Ordnance Bureau is 13000. This is generally written 13000 tons, meaning that 13000 is the ratio of the stress, in tons (of 2240 lbs.) per square inch, to the elongation per inch in inches. For stresses in pounds, the corresponding value of $E$ is 29,120,000 pounds.

The Modulus of Elasticity of Cast Iron is about 8000 tons.

If a metal be loaded within its elastic strength and the load be removed it will resume its original dimensions.

If a metal is loaded beyond its elastic strength the elastic limit of strain will be exceeded and it will not return to its original dimensions. In this latter case *set* or permanent deformation is said to have taken place; the amount by which its dimensions, after the removal of the load, differ from its original or natural dimensions being a measure of the *set*.

When solids are strained far beyond their elastic limits, they assume a condition called Plastic in which they have properties analogous to those of fluids.

In designing a gun it is intended to give it such proportions that, whether the gun is at rest or whether the maximum powder pressure is acting, no part shall be strained beyond its elastic limit, and,

therefore, the result sought in all computations is the Elastic Strength.

It is to be noted that the tensile or breaking strength of steel being greatly in excess of the elastic strength, and the total tensile strain before rupture being from 15% to 20% of the original length, no sound built up gun is likely to burst until the elastic strength has been exceeded for many rounds and the permanent deformation is very apparent.

To this point, in the consideration of the elastic limit of the strain and of the stress producing it, the strain in only one direction has been touched upon, viz., that parallel to the direction of the force applied.

A solid when subjected to a stress in any given direction is not only strained in that direction, but also in the two directions at right angles to the first.

In order to ascertain the relative distribution of strain throughout the mass of a solid, it is necessary to assume that the constitution and properties of the solid are the same at all points and in all directions around all points.

By experiment it has been found, that, when a metal is strained in any direction, within its elastic limit, the strains in the two directions at right angles to the first are each $\frac{1}{3}$ of the strain in the first direction.

This value $\frac{1}{3}$ has been adopted as the best of several numerical values which have been assigned to this quantity by various authorities.

If then a stress $X$ be applied to a unit cube, the strains in the directions of $X$, and in the two directions at right angles to $X$ are as follows

$$\begin{aligned} E_x &= \frac{X}{E}, \\ E_y &= -\frac{X}{E} \cdot \frac{1}{3}, \qquad (1) \\ E_z &= -\frac{X}{E} \cdot \frac{1}{3}. \end{aligned}$$

*Example:* If a steel bar 10 inches long and of square cross section of one square inch area is subjected to a stress of tension of 40,000 pounds in the direction of its length, and if $E$, the modulus of elasticity of the steel is 13000 (tons) or 29,120,000 (pounds) the dimensions of the bar while under the above stress are in length 10.014 inches, and 0.9953 inches in the other two directions, the bar supposed not strained beyond the elastic limit.

Since, within the elastic limit, the strains resulting from any stress is independent of prior stress, if three forces $X$, $Y$ and $Z$ are all acting in the direction of the edges of a unit cube, for example, we shall have from equations (1) for the three strains due to the combined forces in the three directions

$$\begin{aligned} E_x &= \frac{1}{E}\left[X - \frac{Y}{3} - \frac{Z}{3}\right] \\ E_y &= \frac{1}{E}\left[Y - \frac{X}{3} - \frac{Z}{3}\right] \qquad (2) \\ E_z &= \frac{1}{E}\left[Z - \frac{X}{3} - \frac{Y}{3}\right] \end{aligned}$$

Thus if a stress is applied to a unit cube in the direction of $X$ and stresses are then applied in the directions of $Y$ and $Z$, the strains due to $X$ alone will be changed and the amount of the change in the direction of $X$ will be $\frac{1}{3}$ of the strains in the directions of $Y$ and $Z$, due to each of those forces acting alone. In the case of the unit cube, see Fig. 1, if $X$ is a stress of tension and $Y$ and $Z$ are compressive stresses, the strains produced in the direction of $X$ by $Y$ and $Z$ will increase the strain in that direction due to $X$ alone. If the intensity of $X$ when acting alone is such that the strain in the direction of $X$ reaches the elastic limit, it is evident that the effect of $Y$ and $Z$, when they also act, will be to increase the strain in the direction of $X$ beyond the elastic limit. Therefore we have the following important conclusion: The *elastic limit* of *strain* in any of the three directions may be produced by a combination of forces each of less intensity than that necessary to strain the metal to the elastic limit in that direction when acting alone.

As an example suppose in Fig. 1, $Y$ and $Z$ to be compressive stresses each equal to 18 tons per square inch, and suppose that the elastic limit of the metal is .0014''. The modulus of elasticity being 13000 (tons). The strains in the direction of $X$ due to $Y$ and $Z$ are each $\frac{1}{3} \cdot \frac{18.2}{13000}$. Therefore the stress of tension which may then be applied in the direction of $X$ and strain the metal to just the elastic limit is $\frac{18.2}{3}$, as follows:

$$13000 \times .0014 = 18.2 = X + \frac{18.2}{3} + \frac{18.2}{3} \therefore X = \frac{18.2}{3},$$

while if $X$ acted alone, the stress necessary to produce a strain of .0014 is 18.2 tons. It has been assumed in the foregoing that the modulus of elasticity for *compression* is the same as that for *tension.*

## § II.

### EQUATIONS OF EQUILIBRIUM OF A THICK HOLLOW CYLINDER.[*]

It is required to establish the equations of equilibrium of a thick hollow cylinder subjected to internal and external fluid pressure, it being assumed that the lengthwise stress is uniformly distributed over the cross-section of the cylinder. Let Fig. 2 represent the cylinder considered. Draw rectangular axes at any point in the mass $O$; taking $t$ in the direction of a tangent to the cross-section, $p$ in the direction of a radius of the cylinder, and $q$ in the direction of the length. By hypothesis $q =$ a constant; and $t$ and $p$ are to be determined.

In order to do this we may consider the equilibrium of any finite portion of the cylinder, treating it, when the figure of equilibrium under the forces acting has been assumed, as a rigid body. Let Fig. 3 represent a cross-section of the cylinder at any point $O$. It is evident from the symmetry of the forces acting, that all natural circles remain circles after strain; and as the figure of equilibrium is by no means so evident near the ends, we shall neglect these parts and fix our attention on parts near the middle of the length of the cylinder, where the condition of symmetry is fulfilled.

Let $P$ and $P'$ be the internal and external fluid pressures to which the cylinder is subjected; let the two circles, which in the figure are drawn closely together, include the thickness of a very thin cylinder at any point; $p$ and $p'$ be the radial pressures on the inside and outside of this very thin cylinder, $t$ the tension in the direction of the tangent to this cylinder and represent the thickness by $\tau$.

---

[*] Cotterell's Mechanics.

If $p'$ acted alone there would be a compressive stress in the direction of the tangent and if $r'$ is the radius of the point at which $p'$ acts, we have, since $\tau$ is small,

$$p'r' = t_1\tau,$$

$t_1$ being a particular value of $t$; if $p$ acted alone, and $r$ is the radius of the point at which $p$ acts, there would be a pull or stress of tension in the direction of the tangent given by

$$p\,r = t_2\tau.$$

$t_2$ representing another value of $t$.

Therefore when $p$ and $p'$ both act we have

$$p'r' - p\,r = -t\tau$$

in which $t$ is reckoned positive when a pull and represents the resultant of stress. Now if $\tau$ be indefinitely diminished it becomes $dr$, and we have

$$\frac{d(pr)}{dr} = -t. \qquad (3)$$

It is evident that we must obtain a second equation connecting $t$ and $p$, since both are to be determined.

From any one of (2) recollecting that $p$ is oppositely named from $t$

$$E\,e_q = q - \frac{t-p}{3}$$

and from this, since by hypothesis the lengthwise stress $q$ and therefore the strain, is uniform, we have

$$-t+p = 3E\,e_q - 3q$$

$$\text{or, } -t+p = -2c_1 \qquad (4)$$

in which $c_1$ is some constant. Substituting the value of $t$ from (3) in (4) we have

$$\frac{d(pr)}{dr} + p = -2c_1$$

$$\text{or, } r\,dp + 2p\,dr = -2c_1\,dr,$$

multiplying by $r$,

$$r^2dp + 2rp\,dr = -2c_1r\,dr.$$

Integrating

$$r^2p = -c_1r^2 + c_2,$$

in which $c_2$ is the constant of integration

$$\therefore p = \frac{c_2}{r^2} - c_1 \tag{5}$$

From (4) and (5)

$$t = \frac{c_2}{r^2} + c_1 \tag{6}$$

Subtracting (5) from (6), and adding (5) and (6), we deduce

$$t - p = \text{a constant,}$$
$$(t + p)\, r^2 = \text{a constant.}$$

These represent what are known as Lamé's laws and which are expressed as follows:

1. At any point in the mass of a thick hollow cylinder the difference of the hoop tension and radial pressure is a constant.

2. The sum, at any point of a thick hollow cylinder, of the hoop tension and radial pressure multiplied by the square of the radius at that point is a constant.

By means of these laws all the formulas for computing the strength of Guns, according to Virgile's Theory may be deduced.

## § III.

## THE VALUE OF THE CONSTANTS $C_1$ AND $C_2$.

To determine the constants in (5) and (6), call the inside radius of the cylinder $R$ and the outside radius $R'$, and let $P$ and $P'$ be the pressures at these points respectively; then we find, by inserting these values in (5), and elimination:

$$\left.\begin{aligned} c_1 &= \frac{PR^2 - P'R'^2}{R'^2 - R^2} \\ c_2 &= \frac{R'^2 R^2 (P - P')}{R'^2 - R^2} \end{aligned}\right\} \tag{7}$$

For the lengthwise stress on the cylinder since it is constant

$$q\pi (R'^2 - R^2) = \pi (PR^2 - P'R'^2)$$

$$\therefore q = \frac{PR^2 - P'R'^2}{R'^2 - R^2} = c_1. \tag{8}$$

*Examples.*

1. Prove that, when the inside and outside pressures are equal, the hoop-tension and radial pressure are equal at all points; also, that the hoop-tension is negative and equal to $-P$.

2. Prove that, if the external is greater than the internal pressure, $t$ is always negative, and is greatest at the inner surface. If $P' > P$, $c_1$ and $c_2$ are both negative by (7); and (6) then proves the proposition.

3. Prove that, if the internal is greater than the external pressure, but less than $P' \frac{2R'^2}{R'^2 + R^2}$, the hoop-tension is everywhere negative and is greatest at the outer surface. This follows at once, by observing that (6) gives for the hoop-tension at the inner surface of the bore, $T$,

$$T = \frac{P(R'^2 + R^2) - 2P'R'^2}{R'^2 - R^2},$$

and by considering the rate of $t$ along $r$, as given from (6).

4. Trace the locus of (5). From (5) and (6) we have

$$t - p = 2c_1.$$

5. Having given the elastic strength of the metal of a gun consisting of a single tube, find the internal pressure to which it may be subjected without any part being subjected to too great stress. The only part endangered is the inner surface, and 6 gives

$$(T_0 + P_0)R_0^2 = (T_1 + P_1)R_1^2,$$
$$T_0 - P_0 = T_1 - P_1;$$

where $T_0$, $P_0$ and $R_0$ are the hoop-tension, pressure and radius at the surface of the bore; and $T_1$, $P_1$ and $R_1$ the same quantities at the outside of the gun. Eliminating the unknown $T_1$, we have

$$P_0 = T_0 \frac{R_1^2 - R_0^2}{R_1^2 + R_0^2} + \frac{2P_1R_1^2}{R_0^2 + R_1^2}.$$

Here $P_1$, if the gun consists of a single tube, is the atmospheric pressure, and is so small that it may be assumed zero; thus the safe pressure is given by the first term in the above equation.

6. Assuming that it is possible to put together a number of concentric tubes to form the walls of a gun, so that when the maximum powder pressure acts within, the inner surface of each tube shall be

at the elastic limit, and thus all parts doing their best work; show how to compute the internal pressure that such a structure will bear. As in 5, we find for the safe internal pressure in the outside tube,

$$P_1 = T_1 \frac{R_2^2 - R_1^2}{R_2^2 + R_1^2};$$

in which $P_1$ is the same as in example 5. From this equation $P_1$ may be found, since $T_1$ is supposed known, and then the equation in 5 gives $P_0$. It is evident that we may proceed in an exactly similar manner whatever be the number of tubes.

7. Trace the locus of equation (6) and show that for all cases of practical importance the hoop-tension at any point in the thickness of a cylinder is greatest at its inside surface.

8. Prove that, if a gun is to be made of two tubes of the same metal so combined as to pull together at the elastic strength when the maximum powder pressure acts, it will be stronger for a given total thickness, if the radius of the surface of contact of the two tubes is a geometrical mean between the inside and outside radius of the gun, than for any other value. As in example 5, the equation which follows may be found

$$P_0 = T \frac{r^2 - R_0^2}{r^2 + R_0^2} + T \frac{R_2^2 - r^2}{R_2^2 + r^2} - \frac{2r^2}{r^2 + R_0^2},$$

in which $r$ has been written for the radius $R_1$. If here, we put $\frac{dP_0}{dr} = o$, we find $r = \sqrt{R_0 R_2}$.

9. Suppose it were possible to build a gun of an infinite number of tubes so shrunk together that when the maximum powder pressure acted the metal should be pulling uniformly across its thickness, it is required to determine the hoop-tension at any point in the mass of such a cylinder, when the internal powder pressure is not acting. When the powder pressure is acting, the tension at any point is given by

$$t_m = \frac{PR}{R' - R} = b, \text{ suppose};$$

being shown in figure 4*a*, which represents a quadrant of a gun, by *CD*. If the cylinder were a simple one, without initial strain, the tension would be given by (6), *AB* in the figure. Now, if *EF* is a line, such that its ordinates, drawn downwards from *CD*, are equal at every point to the ordinates of *AB*, drawn upwards from *RR'*, then it is obvious that the law of formation of hoop-tension will be fulfilled if, when the powder pressure is created, the line of tension changes from *EF* to *CD*. Therefore *EF* is the line of tensions when the gun is at rest. This condition may be expressed, if $t'$ is the tension at any point when at rest,

$$t' = -\left(\frac{c_2}{r^2} + c_1 - b\right); \text{ (see Fig. 4a.)}$$

10. Prove, from example 12, that $t' = 0$ when $r = \sqrt{RR'}$; that $t' = -\frac{PR'}{R' + R}$ when $r = R$; that $t' = \frac{PR}{R' + R}$ when $r = R'$. Also verify the equation $\int_R^{R'} t' dr = 0$; a condition which obviously should hold.

11. Prove, that, if the condition of example 12 could be realized (which is the perfect fulfillment of the method of building guns on the principle of initial strain), if $T$ be the elastic strength of the metal, and if its elastic strength to resist compression be the same as that to resist tension, the gun cannot be made to bear safely a greater internal pressure than

$$T \times 1.414.$$

If we put the elastic strength of the metal $T$ for $t_m$ in the equation of example 12, we have

$$P = T\frac{R' - R}{R}$$

for the safe pressure. But it is evident from the expression for $t'$ that such values of the radii may be chosen (or the gun so thick) as to cause the metal at the surface of the bore to be compressed to a dangerous extent when the gun is at rest. To discover the greatest thickness which can be used, put $t' = -T$ and $r = R$; then we find upon inserting the values of the constants, $P'$ being zero:

$$-T = \frac{PR'R - PR'^2}{R'^2 - R^2} = \frac{-PR'}{R' + R}.$$

Thus since the internal safe pressure $P$ is given above, we find a relation between the radii which must be fulfilled; it is

$$R' = R(1 \pm \sqrt{2}).$$

This proves the proposition.

## § IV.

## COMPOUND ELASTIC STRENGTH.

Formulas for the Elastic Strength of Guns deduced from Lamé's laws, of which Virgile's formulas are examples, are based on the assumption that the hoop tension, or pull in the direction of the circumference, may be allowed to reach a value equal to the elastic strength of the metal as determined by a free test of specimens in a testing machine, when subjected to a stress of tension alone; the effect of other forces acting in the other two directions, which is the actual condition in the case of a thick tube, being neglected.

Thus in the case of a thick tube subjected to interior pressure alone, we have by Virgile's formulas for the safe interior pressure

$$P = T \frac{R'^2 - R^2}{R'^2 - R^2},$$

in which $R'$ and $R$ are the external and internal radii and $T$ the elastic strength of the metal as determined by free test.

In Fig. 4*b*. If $AB$ is a portion of a thin layer of metal to which the powder pressure imparts a *strain* of hoop-tension equal to the elastic limit in the direction $AB$, it is evident that the resistance of the layers outside of $AB$ will cause a radial pressure perpendicular to $AB$, the effect of which will be to increase the strain in the direction of $AB$, and if, as has been supposed, the interior pressure is of sufficient intensity to strain the metal in the direction of $AB$ to its elastic limit when acting alone, and the strain due to radial pressure is also considered, it is evident that the metal will be strained beyond its elastic limit. Thus we conclude that the *elastic limit of strain* due to the *combined forces*, which act in the case of thick tubes, should be the limiting quantity instead of the tensions or pressures acting alone, and this is the Theory of Major Clavarino, of the Italian Artillery, who first advanced it. The formulas which follow are based on this theory and are due to the authority quoted.

Resuming equations (2); calling $t$, $p$ and $q$ respectively the hoop-tension, radial pressure, and lengthwise stress; and $e_t$, $e_p$, $e_q$ the strains in these directions respectively, we have, for the equilibrium of a thick hollow cylinder subjected to internal and external pressures:

$$\left.\begin{aligned} E_t &= \frac{1}{E}\left[\quad t + \frac{p}{3} - \frac{q}{3}\right] \\ E_p &= \frac{1}{E}\left[-p - \frac{t}{3} - \frac{q}{3}\right] \\ E_q &= \frac{1}{E}\left[\quad q - \frac{t}{3} + \frac{p}{3}\right] \end{aligned}\right. \qquad .(9)$$

remembering that $t$ and $p$ are oppositely named, i. e., $t$ (and $q$) is a pull and $p$ a push,

also

$$\left.\begin{aligned} t &= \frac{c_2}{r^2} + c_1 \\ p &= \frac{c_2}{r^2} - c_1 \\ q &= c_1 \end{aligned}\right\} \qquad (10)$$

Substituting the values of $t$, $p$ and $q$ from (10) in (9) and reducing, we have

$$\left.\begin{aligned} E_t &= \frac{1}{E}\left[\frac{c_1}{3} + \frac{4}{3}\frac{c_2}{r^2}\right] \\ E_p &= \frac{1}{E}\left[\frac{c_1}{3} - \frac{4}{3}\frac{c_2}{r^2}\right] \\ E_q &= \frac{1}{E}\left[\frac{c_1}{3}\right] \end{aligned}\right\} \qquad (11)$$

but, $c_1 = \dfrac{PR^2 - P'R'^2}{R'^2 - R^2}$, and $c_2 = \dfrac{R'^2R^2(P-P')}{R'^2 - R^2}$; (12)

$\therefore$ Equations (11) become, after substitution,

$$E_t = \frac{1}{E}\left[\frac{1}{3}\cdot\frac{PR^2 - P'R'^2}{R'^2 - R^2} + \frac{4}{3}\cdot\frac{R'^2R^2(P-P')}{R'^2 - R^2}\cdot\frac{1}{r^2}\right],$$

$$E = \frac{1}{E}\left[\frac{1}{3}\cdot\frac{PR^2 - P'R'^2}{R'^2 - R^2} - \frac{4}{3}\cdot\frac{R'^2R^2(P-P')}{R'^2 - R^2}\cdot\frac{1}{r^2}\right],$$

$$E_q = \frac{1}{E}\left[\frac{1}{3}\cdot\frac{PR^2 - P'R'^2}{R'^2 - R^2}\right].$$

The strains per unit length in the direction of the circumference, along the radius and in the direction of the length of the cylinder, are represented by $E_t$, $E_p$ and $E_q$ respectively as determined by the preceding equations.

The strain per unit length in the direction of the circumference may be expressed in terms of the radius; if $\Delta c$ denotes the whole strain, since the strain is uniform around the circumference, and $c = 2\pi r$ we have for the strain per unit length

$$\frac{2\pi\Delta r}{2\pi r} \therefore E_t = \frac{\Delta r}{r}.$$

For the strain per unit length at the interior, using the notation adopted, we have $\frac{\Delta R}{R}$, and for the strain per unit length at the exterior $\frac{\Delta R'}{R'}$.

The strain along the radius is not uniform because by (5) and (6) $t$ and $p$ vary for the same internal and external pressures inversely as $r^2$, and if $E_t = \frac{\Delta r}{r}$ we see by inspection that $E_p$ may be derived from $E_t$ by differentiation, or $E_p = \frac{d(\Delta r)}{dr}$. If $h$ is the length of the cylinder, the strain per unit length may be expressed by $\frac{d(\Delta h)}{dh}$.

The equations for the three strains per unit length in the three directions then become

$$\frac{\Delta r}{r} = \frac{1}{E}\left[\frac{1}{3}\cdot\frac{PR^2 - P'R'^2}{R'^2 - R^2} + \frac{4}{3}\cdot\frac{R'^2R^2(P - P')}{R'^2 - R^2}\cdot\frac{1}{r^2}\right], \quad (13)$$

$$\frac{d(\Delta r)}{dr} = \frac{1}{E}\left[\frac{1}{3}\cdot\frac{PR^2 - P'R'^2}{R'^2 - R^2} - \frac{4}{3}\cdot\frac{R'^2R^2(P - P')}{R'^2 - R^2}\cdot\frac{1}{r^2}\right], \quad (14)$$

$$\frac{d(\Delta h)}{dh} = \frac{1}{E}\left[\frac{1}{3}\cdot\frac{PR^2 - P'R'^2}{R'^2 - R^2}\right]. \quad (15)$$

The strains in the direction of the circumference and along the radius at any point of radius $r$ may be found by substituting the proper value of $r$ in (13) and (14).

The limiting value of the strains allowable for any point however, and under any condition whatever, to avoid permanent deformation, *must not exceed the elastic limit of the metal for extension or for compression as the case may be.* These limits are $\frac{\theta}{E}$ and $\frac{\rho}{E}$ respectively, in which $\theta$ and $\rho$ represent the *stress* per unit area which will ***strain*** the metal to its ***elastic limit*** and are determined in practice by free tests of the metal in testing machines.

## *Examples.*

1. Show that the hoop-strain at the surface of the bore of a gun is greater, if $P$ is much greater than $P'$, than it is at any other point.

2. Prove that the contraction of thickness of a hollow cylinder in which $R' = 3R$, subjected to an internal pressure $P$ and *zero* outside pressure, is

$$\frac{P}{E} \cdot \frac{11}{12} \cdot R.$$

3. Prove that the strain per unit volume of a cylinder of radii $R'$ and $R$, subjected to pressures $P'$ and $P$, is

$$\frac{1}{E} \cdot \frac{PR^2 - P'R'^2}{R'^2 - R^2}.$$

The volume strain is $e_t + e_p + e_q$.

4. Prove that, if the internal and external pressures are equal, the strains in the three directions are equal, and are each

$$-\frac{1}{3} \cdot \frac{P}{E}.$$

5. Show that a gun tube, one-half caliber thick, subjected to an external pressure equal to one-fourth of its internal pressure, is not strained in the direction of its length.

This follows at once from the 3d of (9).

6. Suppose that, in a built-up gun, the inner tube is of radii $R' = 2R$; and that when it is subjected to the maximum powder pressure, the external pressure is one-fourth the latter; find the safe internal pressures as given by Virgile's and Clavarino's formulae, the modulus of the metal being 30,000,000, and the elastic limit $\frac{1}{750}$.

Virgile's formula, see example 7, chapter I, gives

$$P = 40{,}000 \text{ pounds}.$$

The result in the preceding example shows that the strain in the direction of the gun's length is zero; thus we have only to fulfil the condition that its hoop-strain shall not exceed the elastic limit. The hoop-strain is given by

$$\frac{\Delta r}{r}=\frac{1}{E}\left[\frac{1}{3}\,\frac{PR^2-P'R'^2}{R'^2-R^2}+\frac{4}{3}\cdot\frac{(P-P')\,R'^2R^2}{R'^2-R^2}\cdot\frac{1}{r^2}\right];$$

this will be greatest when $r$ is least; putting therefore $r=R$, and inserting the values given, we have

$$\frac{\Delta R}{R}=\frac{4}{3}\cdot\frac{P}{E};$$

and, substituting $\frac{\Delta R}{R}=\frac{1}{750}$ and $E=30{,}000{,}000$,

$$P=30{,}000 \text{ pounds.}$$

# CHAPTER II.

## THE ELASTIC STRENGTH OF SIMPLE HOLLOW CYLINDERS.

### § I.

In the investigation of this subject three cases may arise:

1st. When the external pressure is zero, or may be so regarded in comparison with the internal pressure.

2d. When the internal pressure is zero.

3d. When both internal and external pressures act.

These cases will be considered separately and in detail.

### FIRST CASE: $P' = 0$.

Equations 13, 14 and 15, become under this supposition

$$\frac{\Delta r}{r} = \frac{1}{E}\left[\frac{1}{3} \cdot \frac{PR^2}{R'^2 - R^2} + \frac{4}{3} \cdot \frac{PR'^2R^2}{R'^2 - R^2} \cdot \frac{1}{r^2}\right] \qquad (16)$$

$$\frac{d(\Delta r)}{dr} = \frac{1}{E}\left[\frac{1}{3} \cdot \frac{PR^2}{R'^2 - R^2} - \frac{4}{3} \cdot \frac{PR'^2R^2}{R'^2 - R^2} \cdot \frac{1}{r^2}\right] \qquad (17)$$

$$\frac{d(\Delta h)}{dh} = \frac{1}{E}\left[\frac{1}{3} \cdot \frac{PR^2}{R'^2 - R^2}\right]. \qquad (18)$$

It is necessary to determine whether these values are positive or negative for varying values of $r$ and which of the three is the greatest numerically.

The equations show that $\frac{\Delta r}{r}$, and $\frac{d(\Delta h)}{dh}$ are always positive, and that $\frac{d(\Delta r)}{dr}$ is always negative whatever may be the value of $r$, and that if the first be regarded as an extension the second is a contraction, also that $\frac{\Delta r}{r}$ is $> \frac{d(\Delta h)}{dh}$. The equations show also that the maximum elongation and the maximum contraction take place at the inner layer, or when $r = R$, we have then for this point

$$\frac{\Delta R}{R} = \frac{1}{E} \cdot \frac{P}{3} \cdot \frac{4R'^2 + R^2}{R'^2 - R^2};$$

$$\frac{d(\Delta R)}{dR} = -\frac{1}{E} \cdot \frac{P}{3} \cdot \frac{4R'^2 - R^2}{R'^2 - R^2}.$$

Now in order that the cylinder shall not be permanently deformed we must have the conditions that both the elongation $\frac{\Delta R}{R}$ and the contraction $\frac{d\Delta R}{dR}$ shall not exceed the elongation and the contraction which correspond to the elastic strengths of tension and compression respectively. These are $\frac{\theta}{E}$ for tension and $\frac{\rho}{E}$ for compression. We evidently have then the two following equations:

$$\frac{1}{E} \cdot \frac{P}{3} \cdot \frac{4R'^2 + R^2}{R'^2 - R^2} = \frac{\theta}{E}, \tag{19}$$

$$\frac{1}{E} \cdot \frac{P}{3} \cdot \frac{4R'^2 - R^2}{R'^2 - R^2} = \frac{\rho}{E}; \tag{20}$$

from which two values of $P$ may be found, which we call $P_\theta$ and $P_\rho$ and which are

$$P_\theta = 3\theta \frac{R'^2 - R^2}{4R'^2 + R^2}, \tag{21}$$

$$P_\rho = 3\rho \frac{R'^2 - R^2}{4R'^2 - R^2}; \tag{22}$$

and of these the smaller value should be used.

The ratio of the two values is

$$\frac{P_\theta}{P_\rho} = \frac{\theta}{\rho} \frac{4R'^2 - R^2}{4R'^2 + R^2}, \tag{23}$$

and since

$$\frac{4R'^2 - R^2}{4R'^2 + R^2} < 1,$$

the conclusion is that whenever $\theta \overline{\overline{<}} \rho$, $P_\theta$ is certainly less than $P_\rho$; hence, instead of deducing both values in order to compare them we may simply take the first or $P_\theta$. The condition $\theta \overline{\overline{<}} \rho$ is usually fulfilled in practice with the metals which are most used in the fabrication of cannon, and the value $P_\theta$ as found from the formula (21) is alone used.

Formula (21) enables us to find the requisite thickness of the walls of a cylinder of a certain metal in order to withstand a given pressure from within without permanent deformation.

Solving for $R'$, we have

$$R' = R\sqrt{\frac{3\theta + P}{3\theta - 4P}} \tag{24}$$

If $G =$ the requisite thickness $= R' - R$, we have

$$G = R\left(\sqrt{\frac{3\theta + P}{3\theta - 4P}} - 1\right) \tag{25}$$

*Examples.*

1. What is the limiting value of the interior pressure which any simple tube will bear without permanent deformation?

Ans. $\frac{3}{4}\theta$, if $\theta \leqq \rho$.

2. If the walls of a 6″ forged steel shell are 1″.5 thick and if the tensile strength of steel is 50 tons per □″, what powder pressure is necessary per □″ to burst the shell?

Ans. 26.7 tons.

3. What internal pressure will produce an elongation per unit length in the direction of the circumference of 0.0015″ in the case of a simple steel tube of Radii 3″ and 6″? The modulus of elasticity being 13000 (tons). Ans. 10.32 tons.

4. If the pressure on the interior of a simple steel tube is $P$ tons, and if the elongations in the directions of the circumference and radius at any point of the thickness are $a$ and $b$ respectively, $R_0$ and $R_1$ the radii and $E$ the modulus of elasticity for steel, show that the strain in the direction of the length of the tube at any point is $\frac{1}{2}(a + b)$.

5. A gun composed of a simple tube of steel of 26 tons elastic strength is subjected to an internal pressure of 18 tons per sq. inch. The radii of the tube being 3″ and 6″, and the value of $E$ 13000 (tons), determine the strains in the three directions at the interior of the tube.

Ans. $\frac{\Delta R}{R} = .0026''$ elongation,

$\frac{d(\Delta R)}{dR} = -.0023$ contraction,

$\frac{d(\Delta h)}{dh} = .000154$ elongation.

6. With the data of example 5 find the strains in the three directions at the exterior of the tube.

Ans. $\frac{\Delta R'}{R'} = .00077$ elongation,

$\frac{d(\Delta R')}{dR'} = -.00046$ contraction,

$\frac{d(\Delta h)}{dh} = .000154$ elongation.

7. With the data of example 5 determine whether the tube will be permanently deformed and determine what interior pressure the tube will stand without permanent deformation.

Ans. $\frac{\theta}{E} = \frac{p}{E}$ which is less than $\frac{\Delta R}{R}$ or $\frac{d(\Delta R)}{dR}$.

Ans. 13.76 tons.

8. If the elastic strength of cast steel is 30,000 lbs. per sq. inch, and if a 3″ gun be constructed of this metal, what must be the least thickness of the walls over the powder chamber, the diameter of which is 3″.25, in order that the gun may support, without being permanently deformed, a powder pressure of 15000 lbs. per sq. inch?

Ans. 1″.41.

9. If the thickness of the walls of the 3″ steel B. L. R. gun over the powder chamber is 2″.33, the elastic strength of the steel being 30,000 lbs. per sq. inch, the diameter of the powder chamber being 3″.25, what powder pressure will the gun bear at the powder chamber without permanent deformation? Ans. 17,944 lbs.

*Note.*—The mean of 5 determinations of the maximum pressure in the 3″ steel B. L. R. at the Naval Academy, using the spiral pressure gauge, weight of projectile 7 pounds, charge 12 ounces, was 11,200 pounds per square inch; hence the factor of safety for this gun, the dimensions being as given in Example 2, and the elastic strength of the steel being taken at 30,000 lbs. per square inch, is 1.6.

10. The inner radius of a simple tube is 4″, the outer radius 12″, the modulus of elasticity is 13000 for stresses in tons. The outside pressure being zero, determine the inner pressure which will cause a contraction of 0″.01 in the thickness of the tube.

Ans. 35.5 tons.

### § II.

### SECOND CASE: $P = o$.

Equations (13), (14) and (15) become, under this supposition,

$$\frac{⅃r}{r} = \frac{1}{E}\left[-\frac{1}{3}\cdot\frac{P'R'^2}{R'^2-R^2} - \frac{4}{3}\cdot\frac{P'R'^2R^2}{R'^2-R^2}\cdot\frac{1}{r^2}\right], \qquad (26)$$

$$\frac{d_1⅃r}{dr} = \frac{1}{E}\left[-\frac{1}{3}\cdot\frac{P'R'^2}{R'^2-R^2} + \frac{4}{3}\cdot\frac{P'R'^2R^2}{R'^2-R^2}\cdot\frac{1}{r^2}\right], \qquad (27)$$

$$\frac{d_1⅃h}{dh} = -\frac{1}{E}\cdot\frac{1}{3}\cdot\frac{P'R'^2}{R'^2-R^2}. \qquad (28)$$

The first two quantities increase, the one negatively and the other positively, as the radius $r$ diminishes, hence the maximum values of each will be reached when $r = R$, and we have

$$\frac{⅃R}{R} = -\frac{1}{E}\cdot\frac{5}{3}\cdot\frac{P'R'^2}{R'^2-R^2}, \qquad (29)$$

$$\frac{d⅃R}{dR} = \frac{1}{E}\cdot\frac{P'R'^2}{R'^2-R^2}, \qquad (30)$$

$\frac{⅃R}{R}$ and $\frac{d⅃h}{dh}$ are contractions, and $⅃R$ is $> \frac{d⅃h}{dh}$;

$\frac{d⅃R}{dR}$ is an elongation.

In order that the cylinder may not be permanently deformed we have the relations

$$\frac{1}{E}\cdot\frac{5}{3}\frac{P'R'^2}{R'^2-R^2} = \frac{\rho}{E}. \qquad (31)$$

and

$$\frac{1}{E}\cdot\frac{P'R'^2}{R'^2-R^2} = \frac{\theta}{E}; \qquad (32)$$

from which we get

$$P'_\rho = \frac{3}{5}\rho\frac{R'^2 - R^2}{R'^2}, \tag{33}$$

and

$$P'_\theta = \theta\frac{R'^2 - R^2}{R'^2}. \tag{34}$$

Comparing these values, since the smaller must be used, it is easily seen that if $\theta > \frac{3}{5}\rho$, $P'_\theta$ will be $> P'_\rho$.

The condition is generally fulfilled in practice with puddled steel, soft cast-steel, bronze, wrought-iron, but it is not the case with cast-iron or very hard cast-steel, especially when tempered. The following formulae then naturally follow, which give the thickness of the walls of a cylinder necessary to enable it to withstand a given pressure from *without*:

$$G_\rho = R\left(\sqrt{\frac{3\rho}{3\rho - 5P'}} - 1\right) \tag{35}$$

$$G_\theta = R\left(\sqrt{\frac{\theta}{\theta - P'}} - 1\right) \tag{36}$$

## *Examples.*

1. What pressure applied to its exterior can the tube of a 6-inch steel B. L. R. withstand at the powder chamber? The diameter of the powder chamber being 7.5 inches, and the thickness of the metal being 1.75 inches. The elastic strength of the steel for compression being assumed equal to 30,000 pounds, and that for tension the same. Ans. 9632 pounds.

2. Suppose that the pressure on the exterior of the tube of the gun in Example 1, at the powder chamber, was 15000 pounds per square inch, what thickness of metal is necessary to enable it to withstand that pressure, assuming as before $\rho = 30{,}000$ pounds per square inch, and $\theta = 30{,}000$ pounds per square inch.

Using formula (35), because $\theta > \frac{3}{5}\rho$. Ans. 4''.44.

3. The radii of a tube of an 8'' steel gun are 4'' and 7'', and an external pressure of 8.2931 tons per square inch is applied. The modulus of elasticity being 13,840 tons. Find the hoop-strain at the interior. Ans. —0.001483.

4. The interior and exterior of the tube of an 8″ steel gun are 4″.75 and 7″.0 respectively, and at rest the pressure produced on the exterior of the tube by the shrinkage of the jacket and hoops is 7.1682 tons per square inch, the modulus of elasticity being 13840 (tons). Find the hoop-strains per linear unit at the exterior and interior of the tube. Ans. Exterior —0.000909,
Interior —0.0016.

5. The elastic strength of cast-steel being 18 tons, determine the proper thickness of the wall of a 6″ shell at its weakest point in order that it may endure an external pressure of 16 tons per square inch, without being permanently deformed.
Ans. $R_1 = 3''$, $R_0 = 1''$.

6. The exterior radius of a cast-iron common shell at its weakest section is 5″, and the interior radius is 2¼″; the elastic strength for compression of the cast-iron being 14 tons, find what pressure can be put on the base without permanently deforming the shell.
Ans. 11.17 tons per square inch.

## § III.

### Third Case: When $P$ and $P'$ both act.

Evidently there may be three relations of value existing between $P$ and $P'$, as follows: $P > P'$, $P < P'$, and $P = P'$.

Taking them in order, when $P > P'$, it is evident that for a value of $r = R$, or at inner surface of the cylinder, the change of form will be greatest whether in the direction of the circumference or in that of the radius.

We have then by (13)

$$\frac{\Delta R}{R} = \frac{1}{E} \cdot \frac{P(4R'^2 + R^2) - 5P'R'^2}{3(R'^2 - R^2)}; \qquad (37)$$

and, if $P(4R'^2 + R^2) > 5P'R'^2$, $\frac{\Delta R}{R}$ is an elongation, otherwise it is a contraction.

From (15),

$$\frac{d,dh}{dh} = \frac{1}{E} \frac{PR^2 - P'R'^2}{3(R'^2 - R^2)}; \qquad (38)$$

which is an elongation or contraction according as $PR^2 >$ or $< P'R'^2$ respectively.

When it is an elongation

$$\frac{\Delta R}{R} > \frac{d_{,}dh}{dh},$$

when it is a contraction

$$\frac{\Delta R}{R} < \frac{d_{,}dh}{dh},$$

the numerator of (37) being always greater than that of (38).

Finally from (14)

$$\frac{d_{,}\Delta R}{dR} = -\frac{1}{E} \cdot \frac{P(4R'^2 - R^2) - 3P'R'^2}{3(R'^2 - R^2)}; \qquad (39)$$

which is always a contraction; since $P > P'$, and $4R'^2 - R^2 > 3R'^2$, the numerator of (39) is always positive.

Comparing this contraction with that which may take place in the direction of the axis of the cylinder or $\frac{d_{,}dh}{dh}$, we find that

$$\frac{d_{,}\Delta R}{dR} > \frac{d_{,}dh}{dh},$$

because $2PR'^2 - (PR^2 + P'R'^2) > 0$

so that the conditions to be established in order that the elastic strength of the cylinder may not be exceeded, are

$$\frac{P(4R'^2 + R^2) - 5P'R'^2}{3(R'^2 - R^2)} = \theta \qquad (40)$$

and

$$\frac{P(4R'^2 - R^2) - 3P'R'^2}{3(R'^2 - R^2)} = \rho. \qquad (41)$$

From which the values of $P_\theta$ and $P_\rho$ are deduced as follows:

$$P_\theta = \frac{3\theta(R'^2 - R^2) + 5P'R'^2}{4R'^2 + R^2}, \qquad (42)$$

$$P_\rho = \frac{3\rho(R'^2 - R^2) + 3P'R'^2}{4R'^2 - R^2}, \qquad (43)$$

and also the formulae for the thickness,

$$G_\theta = R\left(\sqrt{\frac{3\theta + P}{3\theta - 4P + 5P'}} - 1\right), \qquad (44)$$

$$G_\rho = R\left(\sqrt{\frac{3\rho - P}{3\rho - 4P + 3P'}} - 1\right). \qquad (45)$$

Formula (43) may be put under the form

$$P_\rho = \frac{R'^2 - R^2}{4R'^2 - R^2}(3\rho - P') + P',$$

and shows that the maximum value of $P'$ is $3\rho$, because if $P' > 3\rho$, $P_\rho$ is less than $P'$, which is contrary to the hypothesis made with regard to $P$ and $P'$, and when $P' = 3\rho$, $P_\rho = P'$, it is evident that the maximum value of $P'$ is $3\rho$.

If in formula (45) we make $P = P' = 3\rho$ we get an indeterminate value of $G_\rho$ which signifies that under that hypothesis the cylinder will always undergo a permanent change of form whether the thickness of its walls be small or great.

Let $P < P'$.

Under this condition a contraction takes place in the direction of the circumference which is a maximum at the inner surface; its value being

$$\frac{\Delta R}{R} = -\frac{1}{E}\frac{5P'R'^2 - P(4R'^2 + R^2)}{3(R'^2 - R^2)}.$$

There is a contraction in the direction of the length of the cylinder, as follows:

$$\frac{d\Delta h}{dh} = -\frac{1}{E}\left(\frac{P'R'^2 - PR^2}{3(R'^2 - R^2)}\right),$$

which is numerically less than the preceding.

Finally, there is a contraction in the direction of the radius when

$$PR^2 - P'R'^2 > -\frac{4R'^2R^2(P - P')}{r^2};$$

the first being a negative quantity and the second a positive quantity. This contraction will be a maximum when the second quantity is least, or when $r = R'$, and the value of

$$\frac{d\Delta R'}{dR'} = -\frac{1}{E}\cdot\frac{P'(4R^2 - R'^2) - 3PR^2}{3(R'^2 - R^2)},$$

which is less than $\frac{\Delta R}{R}$.

Consequently, if contraction takes place, it will occur at the inner surface in the direction of the circumference. If elongation takes place, it will occur when in the formula for $\frac{d,\Delta r}{dr}$; the numerator of the first term is less than

$$\frac{4R'^2R\,(P-{}^2P')}{r^2},$$

and this elongation is evidently greatest when $r=R$, or

$$\frac{d\Delta R}{dR}=\frac{1}{E}\,\frac{3P'R'^2-P(4R'^2-R^2)}{3\,(R'^2-R^2)}.$$

The conditions then to be established for determining the maximum external pressure will be

$$\frac{5P'R'^2-P(4R'^2+R^2)}{3\,(R'^2-R^2)}=\rho, \tag{46}$$

and

$$\frac{3P'R'^2-P(4R'^2-R^2)}{3\,(R'^2-R^2)}=\theta. \tag{47}$$

Proceeding as before we obtain

$$P'_\rho=\frac{3\rho\,(R'^2-R^2)+P(4R'^2+R^2)}{5R'^2}, \tag{48}$$

$$P'_\theta=\frac{3\theta\,(R'^2-R^2)+P(4R'^2-R^2)}{3R'^2}; \tag{49}$$

and

$$G_\rho=R\left(\sqrt{\frac{3\rho-P}{3\rho-5P'+4P}}-1\right), \tag{50}$$

$$G_\theta=R\left(\sqrt{\frac{3\theta+P}{3\theta-3P'+4P}}-1\right). \tag{51}$$

Formula (48) may be written

$$P'_\rho=\frac{4R'^2+R^2}{5R'^2}\,(P-3\rho)+3\rho, \tag{52}$$

and shows, as might have been expected from the preceding, that the maximum value of $P=3\rho$, and in that case $P'=P=3\rho$.

Let $P=P'$. We know from the preceding that the maximum value of $P=3\rho$. The fibers of the cylinder are contracted equally in all directions, as will be evident from an inspection of formulae (13), (14) and (15), and is evidently equal to

$$\frac{1}{E}\cdot\frac{PR^2-P'R'^2}{3(R'^2-R^2)}=\frac{1}{E}\,\frac{P}{3}.$$

The relation $\frac{P}{3}=\rho$ being established,

whence $P=3\rho$;

hence, when a cylinder is subjected to two pressures, one internal and the other external, the intensity of those pressures may be made three times the elastic strength of the metal for compression.

When, however, a cylinder is subjected to internal pressure only, and if the thickness of the walls be infinite, the maximum value of $P$ is

$$P = \frac{3}{4}\theta, \text{ or } P = \frac{3}{4}\rho,$$

hence, the advantage gained by subjecting the cylinder to an external pressure at the same time that the internal pressure acts is that the pressure may rise to $3\rho$. And the increase that may be made in the pressure is equal to

$$3\rho - \frac{3}{4}\theta, \text{ or } 3\rho - \frac{3}{4}\rho;$$

this will be the case not only when $\rho > \theta$, but also when $\rho < \theta$ provided that

$$3\rho > \frac{3}{4}\theta, \text{ or } \rho > \frac{1}{4}\theta,$$

and this is always known to be the case whatever may be the thickness of the cylinder.

The hooping and tubing of cannon is designed to increase their elastic strength by producing a pressure on the outside of the cylinder that is subjected to internal pressure, and this internal pressure might attain to the value $3\rho$, if it were not for the condition that each cylinder must be able to resist the external pressure when the internal pressure ceases to act or is less than the external pressure.

# CHAPTER III.

## THE ELASTIC STRENGTH OF BUILT-UP GUNS, COMPOSED OF TWO OR MORE CYLINDERS.

### § I.

### CALCULATION OF THE ELASTIC STRENGTH OF BUILT-UP GUNS WHEN ALL THE ELEMENTARY CYLINDERS WORK TO THE ELASTIC LIMIT.

A Built-up Gun is formed by the superposition of a certain number of simple cylinders, of certain proportions of length and thickness and which may be termed, in discussing the formulas which follow, the *elementary cylinders.*

The object of making a composite cylinder is to call into play the contractile power of the outer cylinders, and thus augment the resistance which can be opposed to permanent deformation by internal pressure.

In superposing the elementary cylinders they may be so placed as to exert, or not exert, an initial pressure upon the innermost cylinder.

The composite cylinder is in the first case said to be under *initial tension,* and in the second case to be without initial tension.

In order that initial tension may exist, it is necessary that the interior diameter of each elementary cylinder, before it is put in place, shall be less than the exterior diameter of the cylinder upon which it is to be superposed, by a certain quantity which is called the *shrinkage.*

It might seem that a composite cylinder without initial compression due to the shrinkage of the elementary cylinders would not give an increase of resistance, which is the principal object of a built-up construction. But it must be remembered that the pressure due to the various elementary cylinders, if it does not exist at first, arises so soon as the internal pressure begins to act; at least this is the case with metals of different elastic powers, as in the case where the cylinders are of different metals. A composite cylinder thus constructed, that is to say, without shrinkage between the elementary cylinders, is termed a composite cylinder of ***variable elasticity.***

In a cylinder under initial tension the initial pressure which is exerted upon the surfaces of contact of the elementary cylinders is increased on the commencement of internal pressure, and increases as the internal pressure increases. A composite cylinder without initial tension is simply a particular case of one combined under shrinkage. The latter case, however, is the one which will be mainly treated.

Atmospheric pressure is neglected because it is very small in comparison with the internal pressure and with those produced by shrinkage.

The composite cylinder may be considered when in two different states.

1st. The state in which the maximum internal pressure acts.

2d. The state of rest, by which is meant that the internal pressure is zero.

In the first state all the cylinders except the outer one are subjected to two pressures, one internal and the other external, and the outer one is under the action of an internal pressure only, the atmospheric pressure being neglected.

In the second state the inner cylinder is subjected to external pressure only, the outer cylinder to internal pressure only, while the intermediate cylinders are subjected to the action of two pressures, one internal and the other external.

In both states, however, the work done by the pressures on any elementary cylinder in causing elongation or contraction must not exceed the elastic limit.

### *Formulas for Elastic Strength.*

Let us now consider the 1st state, or when the maximum pressure acts.

We assume the elementary cylinders to be assembled in a proper manner so that when the maximum pressure acts, each of the elementary cylinders, at its interior surface, will work to the elastic limit.

Resuming equations (13) and (14), which for convenient reference are here re-written:

$$\frac{\Delta r}{r} = E_t = \frac{1}{E}\left[\frac{1}{3} \cdot \frac{PR^2 - P'R'^2}{R'^2 - R^2} + \frac{4}{3} \cdot \frac{(P - P')R'^2R^2}{R'^2 - R^2} \cdot \frac{1}{r^2}\right]$$

$$\frac{d(\Delta r)}{dr} = E_p = \frac{1}{E}\left[\frac{1}{3} \cdot \frac{PR^2 - P'R'^2}{R'^2 - R^2} - \frac{4}{3} \cdot \frac{(P - P')R'^2R^2}{R'^2 - R^2} \cdot \frac{1}{r^2}\right]$$

Equations which represent the *strain* in the direction of the circumference and along the radius respectively, at any point of radius $r$ of the thickness in the case of a simple cylinder of radii $R$ and $R'$, acted upon by the internal pressure $P$ and the external pressure $P'$. We have seen Chapter II, § III, that when $P > P'$, as is the case when the maximum pressure acts in a gun, that these strains are greatest when $r = R$, and (13) and (14) become

$$\begin{aligned} \frac{1}{E} \, \frac{P(4R'^2 + R^2) - 5P'R'^2}{3(R'^2 - R^2)} &= \frac{\theta}{E} \\ \frac{1}{E} \, \frac{P(4R'^2 - R^2) - 3P'R'^2}{3(R'^2 - R^2)} &= \frac{\rho}{E} \end{aligned} \tag{53}$$

if we put for the maximum strain allowable in the direction of the circumference $\frac{\theta}{E}$, and $\frac{\rho}{E}$ for the contraction allowable along the radius.

When these equations are solved for $P$, two values result

$$\begin{aligned} P_\theta &= \frac{3\theta(R'^2 - R^2) + 5\rho'R'^2}{4R'^2 + R^2} \\ P_\rho &= \frac{3\rho(R'^2 - R^2) + 3\rho'R'^2}{4R'^2 - R^2} \end{aligned} \tag{54}$$

and of these the smaller represents the internal pressure which must be adopted.

As has been stated, when the gun is in a state of action each of the elementary cylinders is subjected to two pressures, one internal and the other external, the outer cylinder being subjected to internal pressure only. At the surfaces of contact the internal pressure of any one cylinder is the external pressure of the cylinder next inside.

Thus symmetrical formulas may be written which will express the internal pressure on any cylinder in terms of its radii and the external pressure on that cylinder, involving also $\theta$ and $\rho$ for the metal composing the cylinder. For example take a point in a gun at which the cross-section will represent 3 cylinders. We have for the internal pressure of the outside cylinder

$$(P_2)_\theta = \frac{3\theta_2(R_3^2 - R_2^2)}{4R_3^2 + R_2^2}, \quad (P_2)_\rho \text{ always greater than } (P_2)_\theta$$

and

$$\left.\begin{array}{l} \left.\begin{array}{l} (P_1)_\theta = \dfrac{3\theta_1(R_2^2 - R_1^2) + 5P_2R_2^2}{4R_2^2 + R_1^2} \\ (P_1)_\rho = \dfrac{3\rho_1(R_2^2 - R_1^2) + 3P_2R_2^2}{4R_1^2 - R_1^2} \end{array}\right\} \text{Second cylinder.} \\ \left.\begin{array}{l} (P_0)_\theta = \dfrac{3\theta_0(R_1^2 - R_0^2) + 5P_1R_1^2}{4R_1^2 + R_0^2} \\ (P_0)_\rho = \dfrac{3\rho_0(R_1^2 - R_0^2) + 3P_1R_1^2}{4R_1^2 - R_0^2} \end{array}\right\} \text{Inner cylinder.} \end{array}\right\} \quad (55)$$

Symmetrical equations for four cylinders can be written in the same manner beginning with the outside cylinder.

Use the value of $P_2$ found in the first equation, in the two equations for $P_1$, and of the two values for $P_1$ use the smaller value to find $P_0$. The smaller of which values will be the maximum pressure allowable for the system in action.

### *Examples.*

1. Find the elastic strength of 6″ B. L. R. at powder chamber under Hoop *A*: $R_3$ 10″.25, $R_2$ 8″.25, $R_1$ 5″.63, $R_0$ 3″.5, $\theta_2$.22.05 tons, $\theta_1$ 21.12 tons, $\theta_0$ 17.55 tons, and $\theta = \rho$.

Ans. $(P_2)_\theta =$ 5.01 tons,
$(P_1)_\theta =$ 13.19 tons;
$(P_0)_\rho =$ 19.89 tons.

2. Find the elastic strength of the 6″ B. L. R. at the forward end of Hoop *D*: $R_2$ 7″.5, $R_1$ 5″.16, $R_0$ 3″.0, $\theta_1$ 24.55 tons, $\theta_0$ 17.55 tons, and $\theta = \rho$.

Ans. $(P_1)_\theta =$ 8.67 tons,
$(P_0)_\rho =$ 16.63 tons.

3. Find the elastic strength of the 6″ B. L. R., Mark III, at the after end of Hoop *E*: $R_2$ 7″.0, $R_1$ 4″.99, $R_0$ 3″.0, $\theta_1$ 24,56 tons, $\theta_0$ 17.55 tons, and $\theta = \rho$.

Ans. $(P_1)_\theta =$ 8.04 tons,
$(P_0)_\rho =$ 15.87 tons.

4. Find the elastic strength of 6″ B. L. R., Mark III, at the forward end of Hoop *E*: $R_2$ $6''.49$, $R_1$ $4''.99$, $R_0$ $3''.0$, $\theta_1$ 24.56 tons, $\theta_0$ 17.55 tons, and $\theta = \rho$. Ans. $(P_1)_\theta = 6.56$ tons,
$(P_0)_\rho = 14.65$ tons.

5. Find the elastic strength of 6″ B. L. R., Mark III, at after end of compression slope: $R_3$ $10''.25$, $R_2$ $8''.25$, $R_1$ $5''.63$, $R_0$ $3''.15$, $\theta_2$ 22.05 tons, $\theta_1$ 21.12 tons, $\theta_0$ 17.55 tons, and $\theta = \rho$.
Ans. $(P_2)_\theta = 5.01$ tons,
$(P_1)_\theta = 13.19$ tons,
$(P_0)_\rho = 20.54$ tons.

6. Find the elastic strength of 6″ B. L. R., Mark III, at the after end of Hoop *F*: $R_2$ $6''.32$, $R_1$ $4''.82$, $R_0$ $3''.0$, $\theta_1$ 26.48 tons, $\theta_0$ 17.55 tons, and $\theta = \rho$. Ans. $(P_1)_\theta = 7.25$ tons,
$(P_0)_\rho = 14.95$ tons.

7. Find the elastic strength of 6″ B. L. R., Mark III, at after end of Hoop *B*: $R_3$ $10''.25$, $R_2$ $8''.25$, $R_1$ $5''.63$, $R_0$ $3''.0$, $\theta_2$ 24.78 tons, $\theta_1$ 21.12 tons, $\theta_0$ 17.55 tons, and $\theta = \rho$.
Ans. $(P_2)_\theta = 5.63$ tons,
$(P_1)_\theta = 13.88$ tons,
$(P_0)_\rho = 21.35$ tons.

8. Find the elastic strength of 6″ B. L. R., Mark III, at forward end of Hoop *F*: $R_2$ $6''.07$, $R_1$ $4''.82$, $R_0$ $3''.0$, $\theta_1$ 26.48 tons, $\theta_0$ 17.55 tons, and $\theta = \rho$. Ans. $(P_1)_\theta = 6.34$ tons,
$(P_0)_\rho = 14.19$ tons.

9. Find the elastic strength of the portion of a chase of a gun, given $R_2$ $5''.9$, $R_1$ $4''.2$, $R_0$ $3''.0$, $\theta$ 17.55 tons, $\theta_1$ 28.84 tons, *E* 13000 tons, and $\theta = \rho$. Ans. $(P_1)_\theta = 9.46$ tons,
$(P_0)_\rho = 15.53$ tons.

10. If the radii $R_3$, $R_2$, $R_1$ and $R_0$ of the 6″ B. L. R., Mark III, at the forward end of Hoop *C* were proportioned in Geometric ratio $R_3 = 9''.63$ and $R_0 = 3''.0$, what are the values of $R_2$ and $R_1$ to use on this hypothesis, and what is the elastic strength of the gun at this point when so proportioned: $\theta_2$ 26.12 tons, $\theta_1$ 24.55 tons, $\theta_0$ 17.55 tons, and $\theta = \rho$.

Ans. $R_3$ $9''.63$, $R_2$ $6''.5$, $R_1$ $4''.43$, $R_0$ $3''.0$.

Compare with results in Ex. 14 $\left\{\begin{array}{l}(P_2)_\theta = 9.57, \\ (P_1)_\theta = 19.74,\ (P_1)_\rho = 19.27, \\ (P_0)_\theta = 27.99,\ (P_0)_\rho = 24.42.\end{array}\right.$

11. Find the elastic strength of the 10″ B. L. R., Mark I, No. 3, at the forward end of Hoop *C*: $R_4$ 17″.85, $R_3$ 15″.35, $R_2$ 12″.5, $R_1$ 10″, $R_0$ 5″, $\theta_3$ 51000 pounds, $\theta_2$ 45000 pounds, $\theta_1$ 34,100 pounds, $\theta_0$ 40,700 pounds, and $\theta = \rho$. Ans. $(P_3)_\theta =$ 8,409 pounds,
$(P_2)_\theta =$ 18,769 “
$(P_1)_\rho =$ 27,710 “
$(P_0)_\rho =$ 46,589 “

12. Find the elastic strength of the 6″ gun, Mark III, at a point just forward of trunnion band, having given the following data: $R_3$ 10″.1, $R_2$ 8″.13, $R_1$ 5″.63, $R_0$ 3″.5, $\theta_2 = \rho_2$ 26.12 tons, $\theta_1 = \rho_1$ 21.12 tons, $\theta_0 = \rho_0$ 17.55 tons. Ans. $(P_2)_\theta =$ 5.93 tons,
$(P_1)_\theta =$ 13.99 tons,
$(P_0)_\rho =$ 21.45 tons.

13. Find the elastic strength of 6″ B. L. R., Mark III, at after end of Hoop *G*: $R_2$ 5″.9, $R_1$ 4″.65, $R_0$ 3″.0, $\theta_1$ 28.84 tons, $\theta_0$ 17.55 tons, and $\theta = \rho$. Ans. $(P_1)_\theta =$ 7.09 tons,
$(P_0)_\rho =$ 14.51 tons.

14. Find the elastic strength of 6″ B. L. R., Mark III, at forward end of Hoop *C*: $R_3$ 9″.63, $R_2$ 7″.78, $R_1$ 5″.16, $R_0$ 3″.0, $\theta_2$ 26.12 tons, $\theta_1$ 24.55 tons, $\theta_0$ 17.55 tons, and $\theta = \rho$.
Ans. $(P_2)_\theta =$ 5.85 tons,
$(P_1)_\theta =$ 15.88 tons,
$(P_0)_\rho =$ 22.52 tons.

15. Find the elastic strength of 6″ B. L. R., Mark III, at forward end of the Hoop *G*: $R_2$ 5″.4, $R_1$ 4″.65, $R_0$ 3″.0, $\theta_1$ 28.84 tons, $\theta_0$ 17.55 tons, and $\theta = \rho$. Ans. $(P_1)_\theta =$ 4.72 tons,
$(P_0)_\theta =$ 12.30 tons.

## § II.

## THE STATE OF THE GUN AT REST.

When the maximum pressure $P_0$ acts on the interior of the inner cylinder, and all the elementary cylinders work to their elastic limit at their interior surfaces, the pressures acting at the surfaces of contact $R_1$ and $R_2$ are $P_1$ and $P_2$ respectively, see Fig. 5, the pressure at each surface of contact acts to expand the interior surface of the outer of the two cylinders and it also acts with equal force to contract the outer surface of the cylinder next inside.

To investigate the condition of the system in a state of rest and thus determine if any part of it in this second state will be strained beyond the elastic limit, we know, in the beginning, that the strain will be greatest at the interior surface of the innermost cylinder in a state of rest, because that is the only cylinder unsupported on the interior.

Let $x_0$, $x_1$, $x_2$ and $x_3$, denote the changes in the several pressures $P_0$, $P_1$, $P_2$ and $P_3$ respectively which occur when the system passes from a state of action to a state of rest.

Then, in a state of rest the pressures at the points $R_0$, $R_1$, $R_2$ and $R_3$ become $P_0 + x_0$, $P_1 + x_1$, $P_2 + x_2$, $P_3 + x_3$, respectively.

Now when the maximum pressure ceases to act a change of dimension takes place at each of the surfaces whose radii are $R_0$, $R_1$, $R_2$, $R_3$, and the new dimensions are shown in Fig. 5 by the dotted lines.

If this change of dimension or strain at the point at which the strain is greatest when the system is at rest, the interior surface of the inner cylinder, exceeds the elastic limit, some change must be made in the dimensions of the design, or metal of qualities suitable to the dimensions chosen must be used.

If in the condition at rest we can determine the pressure which acts on the exterior of the inner cylinder by case II, Chap. II, we may determine whether or not permanent deformation will take place at the interior of the inner cylinder. This pressure is $P_1 + x_1$ and found, as follows.

The change of dimension or strain at each point is due alone to the changes in the pressures in passing from one state to the other or to $x_0$, $x_1$, $x_2$ and $x_3$. Therefore we have only to express these strains in terms of the forces which produced them and the corresponding Radii of the particular cylinder whose change of dimension we are considering.

Formula (13) gives the value of the strain per unit length, and for the strains at the interior and exterior surface of a cylinder for which $r = R$ and $r = R'$ respectively, we have

$$\frac{\Delta R}{R} = \frac{1}{E} \cdot \frac{P(4R'^2 + R^2) - 5P'R'^2}{3(R'^2 - R^2)}, \tag{a}$$

$$\frac{\Delta R'}{R'} = \frac{1}{E} \cdot \frac{5PR^2 - P'(4R^2 + R'^2)}{3(R'^2 - R^2)}, \tag{b}$$

in terms of the radii of a cylinder, and the internal and external pressures $P$ and $P'$ which act to produce the strains.

In the present case we wish to find $x_1$, since $P_1 + x_1 = P'_1$, the force which acts on the outer surface of the inner cylinder to produce the maximum strain at $R_0$ when the system is at rest.

The strain at $R_1$ of the outer surface of the inner cylinder is the result of the action of two forces, $x_0$ and $x_1$, and this strain or change of dimension is also the change of dimension of the inner surface of the second cylinder at $R_1$, because in passing from one state to the other the surfaces remain in contact. Therefore formula (b) will be used to represent the change of dimension of the inner cylinder at $R_1$ and formula (a) the change in dimension of the second cylinder at $R_1$, and since the strains are equal we may equate the two expressions, writing $x_0$ and $x_1$ for $P$ and $P'$, and the radii of the inner cylinder in formula (b), and writing $x_1$ and $x_2$ for $P$ and $P'$, and the radii of the second cylinder in formula (a); thus, if $E_0 = E_1$

$$\frac{5x_0 R_0^2 - x_1(4R_0^2 + R_1^2)}{R_1^2 - R_0^2} = \frac{x_1(4R_2^2 + R_1^2) - 5x_2 R_2^2}{R_2^2 - R_1^2}.$$

In the same manner, for the strains at $R_2$, we use again formulas (b) and (a) writing for $P$ and $P'$ the forces which act to produce the change in each cylinder in contact at $R_2$ in passing from one state to the other and the radii of the cylinder considered; thus, equating the expressions

$$\frac{5x_1 R_1^2 - x_2(4R_1^2 + R_2^2)}{R_2^2 - R_1^2} = \frac{x_2(4R_3^2 + R_2)}{R_3^2 - R_2^2}.$$

The first member represents the strain at the outer surface of the second cylinder, and the second member the strain at the inner surface of the third cylinder in which, since the external or atmospheric pressure, $P_3$, is always neglected, $x_3$ is zero. $E_0 = E_1 = E$ for a gun composed of steel tubes.

In the two equations, thus formed we have three quantities, $x_0$, $x_1$ and $x_3$ to determine, but we know the value of $x_0$, since when the system is at rest $P_0 + x_0 = 0$, $\therefore\ x_0 = -P_0$, and there remain but two unknown quantities, $x_1$ and $x_2$, to determine from two equations. From the second we find $x_2 = cx_1$, say, and this value of $x_2$, with $x_0 = -P_0$, when substituted in the first equation allows $x_1$ to be determined.

The pressure acting at the point $R_1$ of the inner cylinder, when at rest, causing contraction or strain at $R_0$ is $P_1 + x_1 = d$, say, and by case II, Chap. II, we have for the safe external pressure allowable on the exterior of a simple tube when the internal pressure is zero the following:

$$P'_\rho = \frac{3}{5}\rho\frac{R_1^2 - R_0^2}{R_1^2}.$$

Therefore if $d > P'_\rho$, the inner cylinder will be strained beyond the elastic limit at the point $R_0$ and the shrinkages which will allow $P_0$ to be adopted as the maximum interior pressure or Elastic Strength of the system in action may not be used. A reduction of the shrinkages, or a modification of the dimensions or qualities of the metals of the separate cylinders, is then necessary, the latter course being generally the simplest and the one usually adopted.

Take the case of an 8″ Gun composed at a certain point of three cylinders. The values of $P_0$, $P_1$ and $P_2$ have been computed and are 47,255, 28,578 and 11,288 respectively. The radii are $R_0 = 5''.25$, $R_1 = 7''.9$, $R_2 = 11''.9$ and $R_3 = 15''.0$; $E_0 = E_1 = E_2$. It is required to find the pressure on the exterior of the inner cylinder, when the gun is at rest, and to determine if this external pressure will permanently deform the bore; we have the following equations:

$$\frac{5x_0 R_0^2 - x_1(4R_0^2 + R_1^2)}{R_1^2 - R_0^2} = \frac{x_1(4R_2^2 + R_1^2) - 5x_2 R_2^2}{R_2^2 - R_1^2},$$

$$\frac{5x_1 R_1^2 - x_2(4R_1^2 + R_2^2)}{R_2^2 - R_1^2} = \frac{x_2(4R_3^2 + R_2^2)}{R_3^2 - R_2^2}.$$

From the latter we deduce a value of $x_2$ in terms of $x_1$ which is found to be $$x_2 = .22601\, x_1.$$

Substituting this value in the former equation, and putting for $x_0$ the value $= -P_0 = -47255$, we find

$$x_1 = -17185.$$

Therefore $$P_1 + x_1 = 11393.$$

If $\theta_0 = \rho_0 = 40{,}000$ pounds, $R_1 = 7''.9$, $R_0 = 5''.25$, we find

$$P'_\rho = \frac{3}{5}\rho\frac{R_1^2 - R_0^2}{R_1^2} = 13{,}407$$

which is the safe pressure allowable on the exterior of the inner tube and is greater than $P_1 + x_1$. Hence the bore will *not* be permanently deformed. A small permanent deformation is however, in practice, not considered disadvantageous.

*Examples.*

1. Determine whether the initial tensions deform the tube permanently in the case of the 6″ B. L. R., Mark III, when the gun is at rest, at the section just forward of the trunnion band: $R_3$ 10″.10, $R_2$ 8″.13, $R_1$ 5″.63, $R_0$ 3″.0, $\theta_2$ 26.12 tons, $\theta_1$ 21.12 tons, $\theta_0$ 17.55 tons, $P_2$ 5.93 tons, $P_1$ 13.99 tons, $P_0$ 21.45 tons, $E$ 13000 (tons).

Ans. $x_2 = 0.2449\, x_1$,
$x_1 = -\,4.6038$ tons,
$P_1 + x_1 = P_1' = 9.39$ tons,
$P_p' = 7.54$ tons,
Tube *will* be permanently deformed.

2. At a certain point in a 6″ gun, for which $R_2 = 5''.9$, $R_1 = 4''.2$, $R_0 = 3''.0$, $\theta_0 = 17.55$ tons, $\theta_1 = 28.84$ tons, $E = 13000$ tons, $P_1 = 9.46$ tons and $P_0 = 15.53$ tons. Determine whether the tube will be permanently deformed when the gun is at rest.

Ans. The tube will stand a pressure $(P_1') = 5.16$ tons,
$x_1 = -\,5.27$ and $P_1 + x_1 = 4.19$ tons
∴ Tube will not be deformed.

## § III.

### THE CALCULATION OF THE SHRINKAGES.

If it has been determined with the dimensions used, and the resulting pressures, that no deformation will occur when the system is at rest, the next step is to determine the shrinkages with which the several parts of the structure are assembled.

The pressures which exist at the surfaces of contact of a system of several cylinders in a state of rest are the results of the shrinkages, or the excess of exterior diameter of an inner cylinder over the interior diameter of the next cylinder outside before the system is assembled. The problem is, then, to determine this excess of diameter at each point of contact, and the key to the solution of the problem is the fact that, when the maximum pressure acts, the inner surfaces of all the cylinders are strained to just their elastic limits,

d, that when the maximum pressure is removed, changes in essures and dimensions take place at each of the surfaces of ntact.

In Figure 6, let the circle of radius *OA* represent a cylinder lready composed of a certain number of elementary cylinders pon which one or more are yet to be placed. Let *OC* be the nterior radius to be placed upon the first. The quantity *AC*, equal to the difference between the two radii *OA* and *OC*, is that which is termed the shrinkage, and which is the quantity to be determined. For the sake of simplicity let us term the cylinder which is to be encircled an *inner* cylinder, and that which encircles it an *outer* cylinder. When the outer cylinder is placed over the inner one, the surface of contact will have a radius *OD*, say, which is greater than *OC* and less than *OA*. The shrinkage is *AC*, or the sum of *CD* and *AD*, in which *AD* represents the contraction undergone by the outer radius of the inner cylinder, and *CD* the extension of the inner radius of the outer cylinder. When other cylinders are put on with shrinkage the surface of contact will take a new position of radius *OE* for example.

The system being supposed completed, if the maximum pressure $P_0$ is applied to the completed structure, the radius of the surface of contact will increase to some point *F*, and in this final condition it is evident that the shrinkage *AC* consists of two parts *FC* and *AF*.

In any system of cylinders assembled with shrinkage let $\varphi_1$, $\varphi_2$, $\varphi_3$, etc., represent the shrinkages per unit length at $R_1$, $R_2$, $R_3$, etc., respectively.

The differences of diameters at these points which must be used in the assemblage of the cylinders are $2R_1\varphi_1$, $2R_2\varphi_2$, $2R_3\varphi_3$, etc., respectively, and these values are for each surface of contact $2A_1C_1 = 2(F_1C_1 + A_1F_1)$, $2A_2C_2 = 2(F_2C_2 + A_2F_2)$, $2A_3C_3 = 2(F_3C_3 + A_3F_3)$, respectively.

In the Figure the quantity *FC* represents the whole strain which the inner surface of the outer cylinder has undergone during all the conditions through which this cylinder has passed, which include its superposition on the inner cylinder the assemblage of other cylinders on its outer surface and its condition under the maximum powder pressure. This total strain, which reaches its maximum when $P_0$ acts, is equal to and must not exceed the elastic limit.

and, that when the maximum pressure is removed, changes in pressures and dimensions take place at each of the surfaces of contact.

In Figure 6, let the circle of radius $OA$ represent a cylinder already composed of a certain number of elementary cylinders upon which one or more are yet to be placed. Let $OC$ be the interior radius to be placed upon the first. The quantity $AC$, equal to the difference between the two radii $OA$ and $OC$, is that which is termed the shrinkage, and which is the quantity to be determined. For the sake of simplicity let us term the cylinder which is to be encircled an *inner* cylinder, and that which encircles it an *outer* cylinder. When the outer cylinder is placed over the inner one, the surface of contact will have a radius $OD$, say, which is greater than $OC$ and less than $OA$. The shrinkage is $AC$, or the sum of $CD$ and $AD$, in which $AD$ represents the contraction undergone by the outer radius of the inner cylinder, and $CD$ the extension of the inner radius of the outer cylinder. When other cylinders are put on with shrinkage the surface of contact will take a new position of radius $OE$ for example.

The system being supposed completed, if the maximum pressure $P_0$ is applied to the completed structure, the radius of the surface of contact will increase to some point $F$, and in this final condition it is evident that the shrinkage $AC$ consists of two parts $FC$ and $AF$.

In any system of cylinders assembled with shrinkage let $\varphi_1$, $\varphi_2$, $\varphi_3$, etc., represent the shrinkages per unit length at $R_1$, $R_2$, $R_3$, etc., respectively.

The differences of diameters at these points which must be used in the assemblage of the cylinders are $2R_1\varphi_1$, $2R_2\varphi_2$, $2R_3\varphi_3$, etc., respectively, and these values are for each surface of contact $2A_1C_1 = 2(F_1C_1 + A_1F_1)$, $2A_2C_2 = 2(F_2C_2 + A_2F_2)$, $2A_3C_3$ $2(F_3C_3 + A_3F_3)$, respectively.

In the Figure the quantity $FC$ represents the whole strain which the inner surface of the outer cylinder has undergone during all the conditions through which this cylinder has passed, which include its superposition on the inner cylinder the assemblage of other cylinders on its outer surface and its condition under the maximum powder pressure. This total strain, which reaches its maximum when $P_0$ acts, is equal to and must not exceed the elastic limit.

If the radius of the surface of contact is $R_1$, $F_1C_1$ is then the whole strain of the interior surface of the second cylinder. When in the completed system $P_0$ acts, there is a pressure $P_1$ at the surface of contact $R_1$ on the interior of the second cylinder, and a pressure $P_2$ at the surface of contact $R_2$ on the exterior of the second cylinder, due to the support of the cylinders outside of the second cylinder. The whole strain $F_1C_1$ is that which would be produced in a cylinder of radii $R_1$ and $R_2$ when acted upon by $P_1$ internally and $P_2$ externally. Similarly $F_2C_2$ is the strain which would be produced at the interior surface of the third cylinder of radii $R_2$ and $R_3$ when the maximum pressure acts due to the pressure $P_2$ on its interior and $P_3$ on its exterior surface, and so on for the value of $F_3C_3$ if there were more than three cylinders in the system.

If then $\frac{F_1C_1}{R_1}$, $\frac{F_2C_2}{R_2}$, $\frac{F_3C_3}{R_3}$, etc., represent the whole strain per unit length on the interior of the cylinders in the final condition, from formula (13), we have

$$\frac{F_1C_1}{R_1} = \frac{1}{E_1}\frac{P_1(4R_2^2 + R_1^2) - 5P_2R_2^2}{3(R_2^2 - R_1^2)}, \text{ for the 2nd cylinder,}$$

$$\frac{F_2C_2}{R_2} = \frac{1}{E_2}\frac{P_2(4R_3^2 + R_2^2) - 5P_3R_3^2}{3(R_3^2 - R_2^2)}, \text{ for the 3rd cylinder,}$$

and so on for the other cylinders.

These strains, since they are elongations, and equal to the elastic limits of the elongations at the elastic strengths of the several cylinders are $\frac{\theta_1}{E_1}$, $\frac{\theta_2}{E_2}$, $\frac{\theta_3}{E_3}$, etc., respectively, if of the two values $P_{1\theta}$, $P_{1\rho}$; $P_{2\theta}$, $P_{2\rho}$, etc., representing the pressure acting at the inner surfaces of the cylinder the proper value is a function of $\theta$.

In practice, however, the assumption is usually made in the case of steel, that $\theta_1 = \rho_1$, $\theta_2 = \rho_2$, etc.

It now remains to determine the value of $AF$, and the deduction will perhaps be best understood by considering the several cases in detail, as must be done in practice.

Suppose we wish to find $A_1F_1$, or that part of the shrinkage yet to be determined at $R_1$.

In Fig. 7. $R_1$ is the radius of contact of the first and second cylinders when the maximum pressure $P_0$ acts at $R_0$, and the pressure $P_1$ acts at $R_1$. If now we remove the pressure $P_0$ and also remove the second cylinder from the first, the outer radius of the first cylinder will increase to $A_1$, because there is now no pressure inside or outside the first cylinder and it will resume its original form.

This increase of radius in passing from a condition of maximum internal pressure and the pressure of the exterior cylinders to that in which there is no internal or external pressure, causes a change of radius $F_1A_1$ which is due to the changes of pressures at $R_0$ and $R_1$ which have taken place in passing from one condition to the other.

Let $y_0$ and $y_1$ represent these changes of pressure in $P_0$ and $P_1$ respectively in passing from the first condition to the second, and by formula (13), writing $r = R_1$, we have for the strain per unit length at $R_1$ due to $y_0$ and $y_1$, in the case of the first cylinder

$$\frac{F_1A_1}{R_1} = \frac{1}{E_0} \cdot \frac{5y_0R_0^2 - y_1(4R_0^2 + R_1^2)}{3(R_1^2 - R_0^2)}.$$

But in the second condition the external and internal pressures are both zero, hence

$$P_0 + y_0 = 0, \text{ and } P_1 + y_1 = 0,$$

or

$$y_0 = -P_0, \text{ and } y_1 = -P_1.$$

Therefore, making these substitutions, we have for the whole shrinkage, per unit length, $\varphi_1$,

$$\varphi_1 = \frac{F_1A_1}{R_1} + \frac{F_1C_1}{R_1} = \frac{1}{E_1} \cdot \frac{-5P_0R_0^2 + P_1(4R_0^2 + R_1^2)}{3(R_1^2 - R_0^2)} + \frac{\theta_1}{E_1}. \qquad (56)$$

The difference of diameter for the surface of contact at $R_1$ is evidently $2R_1\varphi_1$.

In the case of $A_2F_2$, or that part of the shrinkage at $R_2$ which is yet to be found, we suppose, as before, the system to be in action with the pressures $P_0$, $P_1$, $P_2$, etc., acting at $R_0$, $R_1$, $R_2$, etc., respectively.

If, now, the pressure $P_0$ be removed and the third cylinder be taken off, leaving the first and second cylinders still shrunk together, the outer radius of the second cylinder will increase by the distance $F_2A_2$, assuming the dimension it had before the third cylinder was put on. This increase of radius is due to changes of pressure which have taken place in $P_0$, $P_1$, $P_2$, in passing from the condition in which $P_0$ acted on the whole system to that in which $P_0$ and the third cylinder have been removed.

Let $z_0$, $z_1$, $z_2$ represent these changes in $P_0$, $P_1$ and $P_2$ respectively, then $F_2A_2$ is the change of radius or strain at $R_2$, due to the work of the pressures $z_1$ and $z_2$ acting internally and externally respectively on the second cylinder. Therefore by formula (13), as before, for the extension at the exterior radius of the second cylinder, we have per unit length

$$\frac{F_2A_2}{R_2} = \frac{1}{E_2} \cdot \frac{5z_1R_1^2 - z_2(4R_1^2 + R_2^2)}{3(R_2^2 - R_1^2)}.$$

We know that $P_2 + z_2 = 0$ in the final condition when the third cylinder is removed, or $z_2 = -P_2$, but $z_1$ is unknown as yet.

To determine $z_1$ consider in Fig. 7 the change in radius at $R_1$ which takes place at the surface of the two cylinders remaining in contact in passing from one condition to the other. By the removal of the maximum pressure and the third cylinder, the surface of contact contracts to some point as $D_1$, the change in the outer radius of the first cylinder being equal to the change of inner radius of the second cylinder, because the surfaces remain in contact.

Expressing the change of outer radius, or strain in the case of the first cylinder, in terms of the forces $z_0$ and $z_1$ which produced the change, and the radii of the cylinder by formula (13), we have, per unit length,

$$\frac{F_1D_1}{R_1} = \frac{1}{E_0} \cdot \frac{5z_0R_0^2 - z_1(4R_0^2 + R_1^2)}{3(R_1^2 - R_0^2)}.$$

This same change of radius, for the inner radius of the second cylinder is expressed also by formula (13) in terms of $z_1$ and $z_2$, the forces which produced the change, and the radii of the second cylinder.

$$\frac{F_1D_1}{R_1} = \frac{1}{E_1} \cdot \frac{z_1(4R_2^2 + R_1^2) - 5z_2R_2^2}{3(R_2^2 - R_1^2)}.$$

Equating the two expressions, since the surfaces remain in contact:

$$\frac{1}{E_0} \cdot \frac{5z_0R_0^2 - z_1(4R_0^2 + R_1^2)}{R_1^2 - R_0^2} = \frac{1}{E_1} \cdot \frac{z_1(4R_2^2 + R_1^2) - 5z_2R_2^2}{R_2^2 - R_1^2}.$$

In this equation, $z_0 = -P_0$, because $P_0 + z_0 = 0$, and again $P_2 + z_2 = 0$, and $z_2 = -P_2$, making these substitutions $z_1$ may be found.

Substituting the value of $z_1$, thus determined in the formula for for $\frac{F_2A_2}{R_1}$, the second shrinkage, $\varphi_2$, becomes

$$\varphi_2 = \frac{F_2A_2}{R_2} + \frac{\theta_2}{E_2}.$$

The difference of diameters is then, $2R_2\varphi_2$.

If the system is composed of more than three cylinders the value of $F_3A_3$ will be found similarly, the changes in pressures due to change from one condition to the other being $u_0$, $u_1$, $u_2$, $u_3$. Of these $u_0 = -P_0$ and $u_3 = -P_3$; $u_2$ and $u_1$ are determined from a consideration of the changes produced by them, together with $u_0$, at the surfaces of contact $R_1$ and $R_2$ and equating the values for the changes at each surface of contact. The values of $u_1$ and $u_2$ thus determined are substituted in the expression for $\frac{F_3A_3}{R_3}$, as in the other cases.

The following Example will illustrate the method of computing the shrinkages.

In an 8″ Gun, at a cross-section through three tubes, we have the following data

$$R_0 = 5''.25,\ R_1 = 7''.9,\ R_2 = 11''.9,\ R_3 = 15''.0;$$

$$\theta_0 = p_0 = 40{,}000 \text{ pounds},\ \theta_1 = p_1 = 42{,}000 \text{ pounds},\ \theta_2 = p_2 = 47{,}000 \text{ pounds};$$

$$P_0 = 47{,}225 \text{ pounds},\ P_1 = 28{,}578 \text{ pounds},\ P_2 = 11{,}288 \text{ pounds}.$$

The modulus of elasticity is for all the tubes 29,000,000 pounds. For the shrinkage between the jacket and tube, at the inner surface of contact, we have,

$$\varphi_1 = \frac{1}{E}\,\frac{5y_0R_0^2 - y_1(4R_0^2 + R_1^2)}{3(R_1^2 - R_0^2)} + \frac{\theta_1}{E};$$

in which, beside the data already given, we have,

$$y_0 = -P_0 = -47255, \text{ and } y_1 = -P_1 = -28578;$$

therefore

$$\varphi_1 = \frac{-5 \times 47255 \times \overline{5.25}^2 + 28576\,[4 \times \overline{5.25}^2 + \overline{7.9}^2]}{29000000 \times 3\,[\overline{7.9}^2 - \overline{5.25}^2]} + \frac{42000}{29000000}$$

$$= -.00052 + .00145 = .00093.$$

This is the shrinkage per unit length of the radius. For the difference of diameters of the two tubes considered, we have,

$$.00093 \times 15.8 = .0147 \text{ inches}.$$

For the shrinkage between the jacket and hoops, at the outer surface of contact, we have,

$$\varphi_2 = \frac{1}{E} \cdot \frac{5z_1R_1^2 - z_2(4R_1^2 + R_2^2)}{3(R_2^2 - R_1^2)} - \frac{\theta_2}{E}.$$

To determine $z_1$ in this equation, we have,

$$\frac{5zR_0^2 - z_1(4R_0^2 + R_1^2)}{R_1^2 - R_0^2} = \frac{z_1(4R_2^2 + R_1^2) - 5z_2R_2^2}{R_2^2 - R_1^2},$$

In this latter equation,

$$z_2 = -P_2 = -11288,$$
$$z_0 = -P_0 = -47255;$$

whence, writing it in the form,

$$A - Bz_1 = Cz_1 \mp D,$$

where

$$A = \frac{-5 \times 47255 \times \overline{5.25}^2}{\overline{7.9}^2 - \overline{5.25}^2}, \qquad C = \frac{4 \times \overline{11.9}^2 + \overline{7.9}^2}{\overline{11.9}^2 - \overline{7.9}^2},$$

$$B = \frac{4 \times \overline{5.25}^2 + \overline{7.9}^2}{\overline{7.9}^2 - \overline{5.25}^2}, \qquad D = \frac{-5 \times \overline{11.9}^2 \times 11288}{\overline{11.9}^2 - \overline{7.9}^2},$$

we find,

$$z_1 = \frac{A + D}{B + C} = -22315.$$

Therefore,

$$\varphi_2 = \frac{-5 \times 22315 \times \overline{7.9}^2 + 11288[4 \times \overline{7.9}^2 - \overline{11.9}^2]}{29000000 \times 3[\overline{11.9}^2 - \overline{7.9}^2]} + \frac{47000}{29000000},$$

$$\therefore\ \varphi_2 = -.0003697 + .00162$$
$$= .00125.$$

For the difference of diameter at this point we have then

$$.00125 \times 23.8 = .0298 \text{ inches}.$$

*Example.* If the interior diameter of the finished hoop of a gun before the parts are assembled is $8''.4$, what will it be when the parts are assembled and the maximum powder pressure acts? $\theta = 28.84$ tons, $E = 13000$ tons. Ans. $8''.4186$ inches.

Compute the shrinkages at a point just forward of the trunnion band of the 6″ B. L. R., Mark III: $R_2$ 8″.13, $R_1$ 5″.63, $R_0$ 3″.0; $\theta_2$ 26.12 tons, $\theta_1$ 21.12 tons, $\theta_0$ 17.55 tons; $E_0 = 13000$ (tons) $= E_1 = E_2$; $P_2$ 5.93 tons, $P_1$ 13.99 tons, $P_0$ 21.45 tons.

Ans. $\varphi_1 \times 2R_1 = 0''.018053$,
$(y_1 = -8.5882$ tons$)$,
$\varphi_2 \times 2R_2 = 0''.030081$.

## § IV.

## THE LONGITUDINAL STRENGTH.

The longitudinal strain is assumed uniform throughout the thickness of that part of the structure which is subjected to it.

In the case of Guns, the condition which obtains when the maximum pressure acts is alone considered, because in that condition, the maximum longitudinal strain occurs. This strain in the direction of the length of the cylinder is, in the case of a gun, due to the pressure on the breech plug, in the case of breech loaders, or on the bottom of the bore, in the case of muzzle loaders, and to the effect of the pressures on the surfaces of the cylinder in which the breech plug houses. The whole strain must not exceed the elastic limit of the cylinder which carries the breech plug.

The expression for the longitudinal strain is per unit length

$$\frac{d(\Delta h)}{dh} = \frac{1}{E}\left[q - \frac{t-p}{3}\right] = \frac{1}{E}\left[q - \frac{t}{3} + \frac{p}{3}\right].$$

The whole pressure on the breech plug due to the maximum pressure is $\pi P_0 R_0^2$, and this is in equilibrium with the stress, multiplied by the area of the weakest cross-section of the cylinder which carries the breech plug,

$$\therefore\ \pi P_0 R_0^2 = q(\pi R_1^2 - \pi R_0^2)\ \therefore\ q = \frac{P_0 R_0^2}{R_1^2 - R_0^2}.$$

If, as is generally the case, the breech plug is housed in the second cylinder

$$q = \frac{P_0 R_0^2}{R_2^2 - R_1^2}.$$

Now
$$t = \frac{c_2}{r^2} + c_1$$
$$-p = -\frac{c_2}{r^2} + c_1$$
$$\frac{t-p}{3} = \frac{2c_1}{3}$$
$$c_1 = \frac{P_1R_1^2 - P_2R_2^2}{R_2^2 - R_1^2}$$ in the case of the second cylinder.

$$\therefore \frac{1}{E}\left[q - \frac{t-p}{3}\right] = \frac{1}{E} \cdot \left[\frac{P_0R_0}{R_2^2 - R_1^2} - \frac{2(P_1R_1^2 - P_2R_2^2)}{3(R_2^2 - R_1^2)}\right]$$

and
$$\frac{d(\Delta h)}{dh} = \frac{1}{E} \cdot \frac{3P_0R_0 - 2(P_1R_1^2 - P_2R_2^2)}{3(R_2^2 - R_1^2)}.$$

If $L$ denotes the longitudinal resistance or elastic strength per unit area,
$$L = E\frac{d(\Delta h_1)}{dh_1} = \cdot \frac{3P_0R_0 - 2(P_1R_1^2 - P_2R_2^2)}{3(R_2^2 - R_1^2)}.$$

If the breech plug is housed partly in the tube and partly in the jacket, we have
$$L = \frac{3P_0R_0^2 - 2(P_0R_0^2 - P_2R_2^2)}{3(R_2^2 - R_0^2)}$$
and the value of $L$ should not be less than the smaller of the elastic strengths of the two parts per unit area.

# CHAPTER IV.

## WIRE WOUND GUNS ELEMENTARY PRINCIPLES.

The very high elastic strength of steel wire or riband as compared to that of steel in masses has led gunmakers to endeavor to use it in some manner for the construction of Guns. The difficulties which have been met with in its use have been mechanical chiefly, and it is extremely probable that steel wire will play an important part in the construction of guns of great power in the near future. The increase of velocity due to increase of length of gun has a practical limit which is very much less than that attainable by an increase of pressure to twice that used at present, for example. For guns of a given calibre and weight, this great increase of power can only be accomplished by a stronger form of construction, using stronger material, and utilizing the principle of initial tensions so far as is possible.

In order to utilize the principle of initial tensions to the fullest extent when wire is used, the wire is wound upon a tube with a tension of winding, which may be uniform for each layer, or which may be made to vary with each layer. In the latter case it is possible to so regulate the tension of winding that, in the completed gun, the tension, when the maximum pressure acts, will be practically the same for each layer of wire, and each layer will do an equal amount of work, a condition which cannot be realized in a gun composed of thick steel tubes.

The latter case will alone be considered because it is the more rational of the two systems of winding.

The following notation will be used:

$R_0$, $R_1$, the interior and exterior radii of the tube, respectively;

$R_1$, $R_2$, the interior and exterior radii of the wire wound portion, respectively;

$P_0$, the maximum interior powder pressure;

$T$, the tension of each layer of wire when $P_0$ acts;

$t_s$, the tension in a simple tube of dimensions $R_0$, $R_2$, when $P_0$ acts;

$t_i$, the initial tension of the wire when the gun is at rest;

$p$, the pressure at any layer of wire of radius $r$ when $P_0$ acts;

$p_s$, the pressure at any point of radius $r$ in the case of a simple tube when $P_0$ acts;

$p_i$, the pressure at any layer of wire of radius $r$ when the gun is at rest.

For the sake of demonstration suppose the gun to be composed of wire alone of very small cross-section, wound with a varying tension, so regulated that when $P_0$ acts, the tension on each layer of wire will be $T$, any assumed value less than the elastic strength of the wire as determined by a testing machine; and, if certain laws in regard to the condition at rest, and under pressure $P_0$, as well as the tension of winding, can be proved true for any point of radius $r$ of the gun composed of wire alone, the laws will obviously be true for any portion of a thickness, $R_1$, $R_2$, say, Fig. 8, which may be then assumed as the part constructed of wire.

Since the tension is uniform for each layer of wire the line $CD = T$, suppose, will represent the line of tensions when $P_0$ acts in this particular case, see Fig. 8.

If the gun had been constructed of wire wound *without initial tension*, its condition would be the same as a simple tube of the same dimensions, and the laws of tension and pressure at any point in the case of simple tubes, without initial tension, would be true for this condition. We have for simple tubes

$$t_s = \frac{c_2}{r^2} + c_1 \qquad (1)$$

and since the atmospheric pressure is neglected, we have for this condition

$$\frac{c^2}{r^2} = \frac{P_0 R_2^2 R_0^2}{R_2^2 - R_0^2} \cdot \frac{1}{r^2}; \text{ and } c_1 = \frac{P_0 R_0^2}{R_2^2 - R_0^2}$$

$$\therefore \; t_s = \frac{P_0 R_2^2 R_0^2}{(R_2^2 - R_0^2) r^2} + \frac{P_0 R_0^2}{R_2^2 - R_0^2}$$

or

$$t_s = \frac{P_0 R_0^2}{R_2^2 - R_0^2}\left(\frac{R_2^2}{r^2} + 1\right) \qquad (2)$$

a form convenient to use.

Similarly

$$p_s = \frac{c_2}{r^2} - c_1$$

or

$$p_s = \frac{P_0 R_0^2}{R_2^2 - R_2^0}\left(\frac{R_2^2}{r^2} - 1\right) \qquad (3)$$

Formulas (2) and (3) represent the tension and pressure respectively at any point of radius $r$ in the case of simple tubes and for the wire gun wound *without initial tension.* For this condition the line $AB$, Fig. 8, is then the line of tensions when $P_0$ acts and the line $p_s R_2$ is the corresponding line of pressures, the latter being laid off from the former by using the law $t - p =$ constant. The value of the constant is $BR_2$, since $p$ is zero at $R_2$.

If in Fig. 8, we lay off from $CD$ toward $OR_2$ the ordinates of the line $AB$, the result will be the line $EF$, and the line $EF$ will be the line of initial tensions, or the tensions at rest for the gun when wound with varying tension, because when $P_0$ acts the line of tensions will rise from $EF$ and become $CD$, and the condition that the tension shall be uniform when $P_0$ acts will be fulfilled.

From the Figure, with the notation adopted, when the gun wound with varying tension of winding is at rest, we have for the initial tension at any point of radius $r$

$$t_l = T - t_s,$$

substituting the value of $t_s$ from (2), we have

$$t_l = T - \frac{P_0 R_0^2}{R_2^2 - R_0^2}\left(\frac{R^2}{r^2} + 1\right) \qquad (4)$$

Similarly if $p$ is the pressure transmitted by $P_0$ in this gun at any point of radius $r$ and if $p_s$ is the pressure at the same point in the case of a simple tube, we have for $p_l$ the initial pressure

$$p_l = p - p_s.$$

But the pressure at any point of radius $r$ is in equilibrium with the sum of all the tensions outside of $r$ and, since the tension is equal at all points, we have

$$pr = T(R_2 - r),$$

or

$$p = \frac{T(R_2 - r)}{r},$$

therefore, putting also for $p_s$ its value from (3)

$$p_l = \frac{T(R_2 - r)}{r} - \frac{P_0 R_0^2}{R_2^2 - R_0^2}\left(\frac{R_2^2}{r^2} - 1\right) \qquad (5)$$

Having found expressions for the *initial tension* and *initial pressure*, at any point, the next step is to find the *tension of winding*. It is evident that in the completed gun the tension at rest or initial tension is at any point the resultant of the tension of winding at that point and the tension due to the pressure of the layers of wire which were put on outside that point, or by the notation adopted, if $t_w$ = tension of winding, and $t_c$ = tension due to pressure outside of $r$,

$$t_l = t_w + t_c,$$
$$\therefore\ t_w = t_l - t_c. \qquad (6)$$

To determine $t_c$ or the tension at any point $r$ due to the pressure of the layers outside that point, suppose all the layers outside of $r$ to be removed and a pressure $p_l$ to be applied (see eq. 5). The structure will remain in equilibrium and if we regard the layer at $r$ as the outside layer of a simple tube of dimensions $R_0$ and $r$, with a pressure $p_l$ at the outside and zero pressure inside, the tension produced in the outside layer of the simple tube will be the same as that of the layer of wire at $r$, in the case we are considering; the tension produced by $p_l$ is then

$$t_c = -p_l \frac{r^2 + R_0^2}{r^2 - R_0^2}\ * \qquad (7)$$

Resuming equation (6) putting for $t_c$ its value from (7), we have

$$t_w = t_l + p_l \frac{r^2 + R_0^2}{r^2 - R_0^2},$$

and putting for $t_l$ and $p_l$ their values as given by (4) and (5), we have for the *tension of winding* at any point in the case of a gun composed of wire alone wound with varying tension

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*This formula can be deduced from Lamé's laws (see Chap. I, Eq.), which are with the notation adopted

$$t - p_l = T_0 - P_0$$
$$(t + p_l) r^2 = (T_0 + P_0) R_0^2$$

but, $P_0 = 0$ in this application of the laws, hence eliminating $T_0$ we get the expression as quoted.

$$t_w = T - \frac{P_0 R_0^2}{R_2^2 - R_0^2}\left(\frac{R_2^2}{r^2} + 1\right)$$
$$+ \left[T\frac{R_2 - r}{r} - \frac{P_0 R_0^2}{R_2^2 - R_0^2}\left(\frac{R_2^2}{r^2} - 1\right)\right]\frac{r^2 + R_0^2}{r^2 - R_0^2}. \quad (8)$$

This expression may be much simplified: performing the operations indicated it becomes

$$t_w = \frac{TR_2(r^2 + R_0^2) - 2TR_0^2}{r(r^2 - R_0^2)} - \frac{2P_0 R_0^2}{r^2 - R_0^2}. \quad (9)$$

As a portion of the gun in reality is composed of a simple steel tube of radii $R_0$ and $R_1$, the formula may be made to apply to the wire wound portion between $R_1$ and $R_2$. The minimum and maximum values of $r$ being $R_1$ and $R_2$ respectively.

If $r = R_2$, it becomes

$$t_w = \frac{T(R_2^2 + R_0^2) - 2TR_0^2}{R_2^2 - R_0^2} - \frac{2P_0 R_0^2}{R_2^2 - R_0^2} \quad (10)$$

or
$$t_w = T - \frac{2P_0 R_0^2}{R_2^2 - R_0^2},$$

which is identical with the value of $t_l$ in (4) when we let $r = R_2$, as it should be.

Formula (9) may be put in a more convenient form for computation, thus

$$t_w = T\left(\frac{R_2(r^2 + R_0^2)}{r(r^2 - R_0^2} - \frac{2R_0^2}{r^2 - R_0^2}\right) - P_0\frac{2R_0^2}{r^2 - R_0^2},$$

but
$$\frac{2R_0^2}{r^2 - R_0^2} = \frac{r^2 + R_0^2}{r^2 - R_0^2} - 1,$$

therefore
$$t_w = \frac{r^2 + R_0^2}{r^2 - R_0^2}\left[T\frac{R_2}{r} - T - P_0\right] + T + P_0. \quad (11)$$

In deducing the formulas for the elastic strength of guns composed of thick tubes, the Theory of Clavarino was followed. It will be noticed that the formulas for the tension of winding, the tension under pressure $P_0$, and the initial tension in the case under consideration have been deduced by dealing with the *tensions* produced by the action of the forces concerned, instead of the *strains* due to these forces according to Clavarino's Theory. While it is true that the effect of the combined forces in producing strain cannot be wholly neglected it may be said that to follow Clavarino's Theory in the case of wire guns is not only a complicated operation, but the result obtained is not in all cases satisfactory, practically, because, for values of $T$ beyond a certain limit, which is much below attainable values of $T$ in practice, it may be shown that Clavarino's Theory demands a negative tension of winding, which is absurd. Again, if the limiting value of $T$ by Clavarino be used, the value of the tension of winding for the interior layer is zero. If then we assume for $T$ a value below the elastic strength of the wire, no great error will be made by using the formulas as deduced from a consideration of the *tensions* produced in the layers of wire.

For the elastic strength of the completed gun we may, however, use Clavarino's Theory as applied to the tube, which supports a pressure $P_0$ within and $P_1$ without.

The value of $P_1$ is

$$P_1 = \frac{T(R_2 - R_1)}{R_1},$$

$$\therefore \quad (P_0)_\theta = \frac{3\theta(R_1^2 - R_0^2) + 5TR_1(R_2 - R_1)}{4R_1^2 + R_0^2}$$

$$(P_0)_\rho = \frac{3\rho(R_1^2 - R_0^2) + 3TR_1(R_2 - R_1)}{4R_1^2 - R_0^2}$$

and of these values the smaller should be used.

In the construction of a wire wound gun we have the imposed condition that the bore must not be permanently deformed when the gun is at rest.

From (5) letting $r = R$.

$$(p_t)_R = \frac{T(R_2 - R_1)}{R_1} - \frac{P_0 R_0^2}{R_2^2 - R_0^2}\left(\frac{R_2^2}{R_1^2} - 1\right).$$

In order that the bore shall not be permanently deformed, we have (Chap. II, case II)

$$(p_t)_{R_1} = \text{ or } < \frac{3}{5}\rho\frac{R_1^2 - R_0^2}{R_1^2}.$$

The limit of the elastic strength of any gun, however constructed, is the sum of the elastic strengths for tension and compression of the inner tube of the system, which in a wire wound gun is the tube on which the wire is wound. The initial pressure allowable on the outside of the tube when the gun is at rest is a function of $T$, the tension assumed for the wire when in action, the pressure $P_0$ which it is desired to attain, and the dimensions of the parts. The tube must be thick enough to sustain the weight of the wire, and the outside casing without flexure, as well as to withstand the compression induced by winding. The breech block is generally housed in a jacket which is placed outside the wire wound body, and the dimensions of the jacket are found by the method indicated in par. IV, Chap. III.

# CHAPTER V.

## BIRNIE'S FORMULAS.

Captain Rogers Birnie, Jr., Ordnance Dep't. U. S. Army, has deduced complete formulas for the "Resistance of Built-up Guns and the Shrinkages adapted to their Construction," and his conclusions are published in full in No. 35, Notes on the Construction of Ordnance. It is proposed to give here, in view of the high authority of the author, and the remarkable agreement of his formulas with the results of experiment for the special and important condition of the gun at rest, which is used as the basis of the deduction of the formulas, so much of Captain Birnie's work as will enable a comparison to be made with results obtained from Clavarino's formulas. The pages which follow are taken, so far as space permits almost verbatim from the Ordnance Note above quoted.

### INTRODUCTORY.

The method employed in the construction of a gun, assembled by the successive shrinkage of two or more cylinders or rows of hoops one upon another, leads to a consideration of the state of rest of such a system as the primary and one of the most important phases of the problem involved in the deduction of the proper shrinkages to be employed.

The accepted theory of this mode of construction is to assemble the several rows of cylinders so that:

In whatever state the system may be considered, none of the fibers of any cylinder in the structure shall be elongated or contracted beyond the elastic limits determined for such displacements by the free tests of the metal.

With the system at rest this applies especially to the tube which, ordinarily, has to support alone, or without other assistance than the atmospheric pressure, the accumulated stress due to the shrinkages of all the outside cylinders. Under these circumstances the surface of the bore undergoes the greatest change of form by compression, so that the shrinkages of the outer cylinders must be limited to retain uninjured the elastic properties of the metal at the surface of the bore of the tube.*

With the system in action, that is, subjected to the maximum interior pressure which it can support with safety, the cylinders or hoops composing each layer of the structure should work together to the elastic limit of their metal. Here again it is the interior fibers which undergo the greatest change of form in general by circumferential extension in the outer cylinders and by radial compression in the inner cylinders. The theoretical resistance of the gun must then be limited to retain uninjured the elastic properties of the metal at the interior of any of the cylinders composing the structure. It involves the following considerations, viz.: As many of the cylinders as practicable should work together to the elastic limit of their metal under extension, but when other cylinders are endangered from radial compression of their walls the theoretical interior pressure must be curtailed to provide against such over compression, and the working tensions of the first-named parts will be correspondingly curtailed. However, the wall of the tube (or part of the structure next to the bore) has always to support the greatest normal pressures with the system in action; hence, frequently, in this state of the system also the theoretical resistance of the gun will be limited by the strength of the tube to resist compression, in this case radial instead of tangential, as in the other extreme state of the system.

With the system at rest, the only *extraneous* forces to be considered are the *normal* pressures which act upon the elementary cylinders at the several contact surfaces and are produced by the laying on of exterior cylinders. When the gun is fired a certain longitudinal traction is developed by the pressure of the gas on the bottom of the bore, which must be considered as one of the forces existing for the state of the system in action. The principal effect of this traction will be to produce an extension of the metal in the direction of the axis, but it will also produce changes in dimensions at right angles to that direction. Hence, although the longitudinal resistance of the structure forms a subject for consideration apart from the tangential resistance, yet the equations of equilibrium relating to the latter must be influenced to a certain extent by the presence of a longitudinal traction. These equations of equilibrium, from which also we derive the shrinkages to be used for the assemblage of the gun, should be satisfied for *any state of the system whatever*. It will be found difficult to make them so, as will appear in the discussion which follows. We will therefore endeavor to prove that the longitudinal tension developed in firing may, without noteworthy error, be neglected in deducing the equations of equilibrium expressing the relations for any state of the system whatever between the elastic forces of the metal which act at right angles to the axis of the bore and the exterior forces acting upon the cylinders. These elastic forces are the tangential and radial resistances which are functions of the shrinkages.

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* It is, perhaps, an open question whether the compression of the bore may not with advantage be carried beyond this limit, but for the purposes of theoretical discussion we assume that it should not be.

In thus neglecting to consider the longitudinal tension, a source of such tension which is indigenous to the construction of a gun assembled by shrinkage under heat and separate from the action of the powder has not been overlooked. This refers to the constrained position which each heated cylinder assumes in cooling; the heat elongates the piece which, therefore, must clamp the cool under cylinder circumferentially whilst still elongated and there is produced a state of longitudinal tension which is maintained by the friction between the two surfaces in contact. The continued cooling of the heated cylinder will exert an effort to contract the length of the cylinder beneath, and the reaction will affect in a contrary manner the cylinder that was heated. Hence it is seen that the longitudinal tension thus produced in building up the system is of a compensatory nature. And so far as our experience goes the theories herewith, which neglect also this source of longitudinal tension by considering only the normal pressures at the contact surfaces in a built-up gun, and that these act as fluid pressures, do give results for both the radial and longitudinal displacements of the cylinders which agree closely with those derived from actual experiment.

Respecting the longitudinal resistance of a built-up gun, that will depend upon the manner in which the several cylinders composing the reinforce are connected or in which the strains produced by the pressure of the powder gas on the bottom of the bore are distributed.

Birnie's formulas are derived as follows: From Chapter I, Equations (9) and (10): If $t$, $p$ and $q$ are respectively the hoop-tension, radial pressure and lengthwise stress; and $E_t$, $E_p$, $E_q$ the strains in these directions respectively, we have for the equilibrium of a simple cylinder closed at both ends and subjected to internal and external pressure

$$\left.\begin{aligned} Ee_t &= t + \frac{p}{3} - \frac{q}{3} \\ Ee_p &= -p - \frac{t}{3} - \frac{q}{3} \\ Ee_q &= q - \frac{t}{3} + \frac{p}{3} \end{aligned}\right\} \qquad (1)$$

and

$$\left.\begin{aligned} t &= \frac{c_2}{r^2} + c_1 \\ p &= \frac{c_2}{r^2} - c_1 \\ q &= c_1 \end{aligned}\right\} \qquad (2)$$

$$c_1 = \frac{PR^2 - P'R'^2}{R'^2 - R^2}, \quad c_2 = \frac{R'^2R^2(P - P')}{R'^2 - R^2}.$$

Now if in these equations we assume that there is no longitudinal traction or that $q = o$, which is the same thing as considering the cylinder as open at both ends, the equations for the strains become

$$\left.\begin{aligned} Ee_t &= t + \frac{p}{3} \\ Ee_p &= -p - \frac{t}{3} \\ Ee_q &= -\frac{t}{3} + \frac{p}{3} \end{aligned}\right\} \qquad (3)$$

in which

$$t = \frac{PR^2 - P'R'^2}{R'^2 - R^2} + \frac{R'^2R^2(P - P')}{R'^2 - R^2} \cdot \frac{1}{r^2} \qquad (4)$$

and

$$p = -\frac{PR^2 - P'R'^2}{R'^2 - R^2} + \frac{R'^2R^2(P - P')}{R'^2 - R^2} \cdot \frac{1}{r^2} \qquad (5)$$

when the value of $c_1$ and $c_2$ have been substituted.

Using the same notation as in the deduction of Clavarino's formulas and substituting the values of $t$ and $p$ in expressions (3) we have

$$\frac{\Delta r}{r} = \frac{1}{E} \cdot \frac{2}{3} \cdot \frac{(PR^2 - P'R'^2)}{(R'^2 - R^2)} + \frac{1}{E} \cdot \frac{4R'^2R^2(P - P')}{3(R'^2 - R^2)} \cdot \frac{1}{r^2}$$

$$\frac{d.\Delta r}{dr} = \frac{1}{E} \cdot \frac{2}{3} \frac{(PR^2 - P'R'^2)}{(R'^2 - R^2)} - \frac{1}{E} \cdot \frac{4R'^2R^2(P - P')}{3(R'^2 - R^2)} \cdot \frac{1}{r^2}$$

$$\frac{d.\Delta h}{dh} = -\frac{1}{E} \cdot \frac{2}{3} \frac{(PR^2 - P'R'^2)}{(R'^2 - R^2)} .$$

from which to determine the relative changes in the direction of the circumference, of the radial thickness of the wall, and of the length of the cylinder.

The limiting values of $\frac{\Delta r}{r}$ and $\frac{d.\Delta r}{dr}$ are $\frac{\theta}{E}$ and $\frac{\rho}{E}$, respectively.

The value of $\frac{d.\Delta h}{dh}$ gives simply the elongation of the fibers in the direction of the length of the cylinder due to the normal pressures $P$ and $P'$.

The general case to be considered is under the supposition that the cylinder is acted upon by both an internal and external pressure.

The limiting values of these pressures will be obtained by considering the displacements of the fibers at the interior surface of the cylinder where for varying value of $P$ and $P'$ the displacements will be equal to or greater than those produced elsewhere. It being understood that such displacements are to be estimated both in the direction of the circumference and of the radius, and that the pressures must be limited so as not to exceed the limiting displacement in either of these directions.

Substituting $R$ for $r$ in equations (6) and (7), we obtain:

$$\frac{\Delta R}{R} = \frac{(4R'^2 + 2R^2)P - 6R'^2P'}{3(R'^2 - R^2)E} \tag{9}$$

$$\frac{d\Delta R}{dR} = -\frac{(4R'^2 - 2R^2)P - 2R'^2P'}{3(R'^2 - R^2)E} \tag{10}$$

which are the general equations for determining the displacements produced at the interior surface of the cylinder under given values of $P$ and $P'$.

If there be no exterior pressure, $P' = o$ and equation (9) becomes

$$\frac{\Delta R}{R} = \frac{(4R'^2 + 2R^2)P}{3(R'^2 - R^2)E}. \tag{11}$$

If $P = o$, we have

$$\frac{\Delta R}{R} = -\frac{2R'^2P'}{(R'^2 - R^2)E}. \tag{12}$$

Similarly for the displacements in the direction of the radius we find from equation (10):

If $P' = o$,

$$\frac{d\Delta R}{dR} = -\frac{(4R'^2 - 2R^2)P}{3(R'^2 - R^2)E}. \tag{13}$$

If $P = o$,

$$\frac{d\Delta R}{dR} = \frac{2R'^2P'}{3(R'^2 - R^2)E}. \tag{14}$$

With $R$ substituted for $r$, the limiting values for the relative changes are expressed by:

$$\frac{\Delta R}{R} = \frac{\theta}{E}, \text{ and } \frac{d\Delta R}{dR} = \frac{\rho}{E}$$

whence from equation (9) we obtain:

$$\frac{(4R'^2 + 2R^2)P - 6R'^2P'}{3(R'^2 - R^2)} = \theta$$

and from (10):

$$\frac{(4R'^2 - 2R^2)P - 2R'^2P'}{3(R'^2 - R^2)} = \rho$$

from which we find two values for $P$, viz.:

$$P^{(1)} = \frac{3(R'^2 - R^2)\theta + 6R'^2P'}{4R'^2 = 2R^2} \tag{15}$$

and

$$P^{(2)} = \frac{3(R'^2 - R^2)\rho + 2R'^2P'}{4R'^2 - 2R^2}. \qquad (16)$$

The least of these must be taken to find the limiting elastic resistance of the cylinder subjected to an interior and an exterior normal pressure, the longitudinal traction being supposed nil. These equations therefore express the value of $P$ consistent with the elastic equilibrium of the cylinder when forming part of the structure of a built-up gun and the system is at rest.

If $P' = o$, we find:

$$P^{(1)} = \frac{3(R'^2 - R^2)\theta}{4R'^2 + 2R^2} \qquad (17)$$

$$P^{(2)} = \frac{3(R'^2 - R^2)\rho}{4R'^2 - 2R^2} \qquad (18)$$

If $P = o$, equations (12) and (14) show that $\frac{\Delta R}{R}$ is a contraction and $\frac{d\Delta R}{dR}$ an extension, the limiting values for these two quantities will then be expressed by

$$\frac{\Delta R}{R} = \frac{\rho}{E} \text{ and } \frac{d\Delta R}{dR} = \frac{\theta}{E}$$

so that the two values for $P'$, the exterior pressure, will be:

$$P'^{(1)} = \frac{(R'^2 - R^2)\rho}{2R'^2} \qquad (19)$$

$$P'^{(2)} = \frac{3(R'^2 - R^2)\theta}{2R'^2} \qquad (20)$$

These last two equations will be found especially useful for determining the limiting exterior pressure that the tube will support in calculating the shrinkages of the gun structure from a consideration of the system in a state of rest; and since we will generally find $\frac{1}{2}\rho < \frac{3}{2}\theta$, equation (19) will generally serve to determine the limiting value.

For the relative circumferential (or radial) change at the exterior surface of the cylinder, we find by substituting $R'$ for $r$ in equation (6):

$$\frac{\Delta R'}{R'} = \frac{6R^2P - (4R^2 + R'^2)P'}{3(R'^2 - R^2)E} \tag{21}$$

If $P' = o$, we find:

$$\frac{\Delta R'}{R'} = \frac{2R^2P}{(R'^2 - R^2)E} \tag{22}$$

If $P = o$, we find:

$$\frac{\Delta R'}{R'} = -\frac{(4R^2 + 2R'^2)P'}{3(R'^2 - R^2)E} \tag{23}$$

Respecting the relative changes in the direction of the length of the cylinder, equation (8) shows that if $P > P'$ there will be a contraction, unless

$$2P'R'^2 > 2R^2P;$$

if $P' = o$, there will always be a contraction, given by:

$$\frac{d\Delta h}{dh} = -\frac{2R^2P}{3(R'^2 - R^2)E} \tag{24}$$

If $P' > P$, or if $P = o$, there will always be an elongation, given in the latter case by:

$$\frac{d\Delta h}{dh} = \frac{2R'^2P'}{3(R'^2 - R^2)E} \tag{25}$$

Applied in the case of a built-up gun, the quantities $P$ and $P'$ in the preceding equations represent simply the interior and exterior normal pressures at the surfaces of any one of the cylinders, induced by the assemblage of the several layers under the shrinkages which may be found applicable to the particular gun and metals in question.

The equations do not express exactly the equilibrium of the system in action, since in that case a certain longitudinal traction is acting. If, then, we make use of these equations to express the equilibrium of the system in *any state whatever*, it can only be claimed that the results will be approximate for the system in action or for any intermediate state short of that of rest.

It then becomes a question: how nearly will the value of $P$, used to express the resistance of the gun, derived from these formulas, approximate to the true value of that resistance? It might also be questioned whether the shrinkages derived from these equations are entirely appropriate, since as the longitudinal traction increases with the powder-gas pressure, the values for the radial (or circumferential) changes will not be correctly expressed by these equations, and the different parts of the structure will not work in accordance with the theory stated at the commencement of this paper. But it is considered that a discussion of the first question will involve both, since the resistance of the gun and the shrinkages are functions one of the other.

It may readily be shown, owing to the variableness of the longitudinal traction and its unequal distribution in the layers of the gun, not only that it would be extremely difficult to find equations expressing the equilibrium of the system in action, but further, that no great error will be committed in adopting the value of the tangential resistance $P$, to be found from the equations already deduced, which omit the longitudinal traction altogether.

The maximum value possible for this traction will be found by assuming that the interior pressure $P$ is fully developed before the shot begins to move ; we would then have a cylinder closed at both ends, with the pressure $P$ acting upon the bases or we would have the same strain in the cross-section through the bottom of the bore if we suppose that the gun has no recoil. Then the total force would be expressed by

$$\pi R^2 P,$$

and calling $R_n$ and $R_{n-1}$ the interior and exterior radii of the cross-section of metal which has to support this traction, upon the supposition that the tension was uniformly distributed throughout the cross-section, we would have :

$$q = \frac{\pi R^2 P}{\pi(R_n^2 - R_{n-1}^2)} = \frac{PR^2}{R_n^2 - R_{n-1}^2}$$

in which $q$ represents the tension per unit of surface.

If this tension were uniformly distributed throughout the cross-section of a built-up gun, this value for $q$ would be applicable for the case of the system in action to each of the layers of metal, and by substituting, as before, for $t$ and $p$ in expressions (1), together with

$$q = \frac{PR^2}{R'^2 - R^2}$$

we would obtain general equations similar to (15) and (16), expressing the relations between the tangential resistance of any cylinder in the gun and the elastic strength of its metal. But, in fact, such equations of equilibrium would be satisfied only for the section of the gun about the chamber, since forward of the shot and nearing the trunnions, there is a decreased and decreasing value for the longitudinal traction. Again, the longitudinal traction is not equal in the several layers.

Take for illustration the case of a steel breech-loading gun composed of a tube, jacket, and two layers of hoops, in which the jacket supports the block. In this gun the longitudinal tension developed in firing will be mainly supported by the jacket cylinder, except for such assistance as it will derive owing to the friction of adjacent surfaces; hence the longitudinal tension is by no means uniformly distributed throughout the layers, especially in the case of the hoops which are disconnected longitudinally. And, therefore, we cannot deduce a general equation involving the value of the longitudinal traction which will be applicable either to any cylinder whatever of the gun, or even to any single cylinder throughout its whole length.

Let us assume, however, that such equations would be satisfied for the jacket cylinder of this gun (they would evidently not be so for the other cylinders), and examine the form of the resulting equations as compared with (15) and (16).

If in expressions (1) we substitute the values for $t$ and $p$ before used, together with

$$q = \frac{PR^2}{R'^2 - R^2}$$

we will obtain:

$$\frac{\Delta r}{r} = \frac{PR^2 - 2R'^2P'}{3(R'^2 - R^2)E} + \frac{4R'^2R^2(P - P')}{3(R'^2 - R^2)E}\,\frac{1}{r^2}$$

and

$$\frac{d\Delta r}{dr} = \frac{PR^2 - 2R'^2P'}{3(R'^2 - R^2)E} - \frac{4R'^2R^2(P - P')}{3(R'^2 - R^2)E}\,\frac{1}{r^2}$$

from which, substituting $R$ for $r$, we obtain:

$$\frac{\Delta R}{R} = \frac{(4R'^2 + R^2)P - 6R'^2P'}{3(R'^2 - R^2)E}$$

and

$$\frac{d\Delta R}{dR} = -\frac{(4R'^2 - R^2)P - 2R'^2P'}{3(R'^2 - R^2)E}$$

whence, with

$$\frac{\Delta R}{R} = \frac{\theta}{E}, \text{ and } \frac{d\Delta R}{dR} = \frac{\rho}{E}.$$

we obtain

$$P^{(1)} = \frac{3(R'^2 - R^2)\theta + 6R'^2P'}{4R'^2 + R^2} \tag{26}$$

$$P^{(2)} = \frac{3(R'^2 - R^2)\rho - 2R'^2P'}{4R'^2 - R^2} \tag{27}$$

These differ from (15) and (16) only in that $R^2$ takes the place of $2R^2$ in the last term of the denominators. So that the value of $P^{(1)}$ is increased and the value of $P^{(2)}$ is decreased. From which it is seen that if the limiting assistance of the block-carrying cylinder is governed by its capacity to resist circumferential extension, the value for $P^{(1)}$ derived from equation (15) would be on the safe side. If, on the other hand, that cylinder is endangered from radial compression of the wall with the system in action, the limiting value $P^{(2)}$ derived from equation (16) would be too great.

It would require the solution of particular examples to deduce the exact difference in these values for $P^{(1)}$ and $P^{(2)}$, but at least in the construction under consideration the difference would be small as affecting the resistance of the whole structure; and it would generally be so, since these new equations would apply only to the block-carrying cylinder. We conclude, therefore, that the equations first deduced should be used for the following reasons:

1st. They express the equilibrium of each cylinder in the system in a state of rest.

2d. With the system in action the results obtained by their application for either of the hoop cylinders are theoretically exact, and nearly so for the tube cylinder, while for the jacket or block-carrying cylinder the results are approximately true. And, considering the system as a whole, the value of the tangential resistance $P$ thus deduced will be a sufficient approximation for all practical purposes to regulate the powder-gas pressure to which the gun may safely be subjected when first fired.

### The Longitudinal Resistance.

This must be considered apart from the tangential resistance. Referring to the gun already named as an example, it appears that the jacket cylinder should in itself possess sufficient elastic resistance to sustain the longitudinal traction developed in firing. The longitudinal resistance of the gun will then depend upon that of this cylinder. We assume in this case also that the extension of the fibers of the metal in the direction of the axis of the cylinder cannot safely be carried beyond the elastic limit of the metal.

To make the demonstration applicable to the block-carrying cylinder in any gun, adopt the following notations:

$P$ = interior pressure in bore of gun.

$P_n$, $P_{n-1}$ = exterior and interior normal pressures acting directly upon the block-carrying cylinder.

$R$ = radius of bore (or chamber) of gun.

$R_n$, $R_{n-1}$ = radii of exterior and interior surface of block-carrying cylinder.

$\theta_{n-1}$ = elastic limit, under extension, of metal per unit of area.

$E_{n-1}$ = constant modulus of elasticity of the metal of cylinder.

It is assumed that the stresses and strains will be uniformly distributed throughout the cross-section of the cylinder.

The pressure $P$, acting when the gun is fired, will produce a maximum longitudinal tension per unit of area in the cross-section of the cylinder, given by

$$q = \frac{PR^2}{R_n^2 - R_{n-1}^2};$$

the normal pressures $P_n$ and $P_{n-1}$ will give [similar to equations (4) and (5)]

$$t = \frac{P_{n-1} R_{n-1}^2 - P_n R_n^2}{R_n^2 - R_{n-1}^2} + \frac{R_n^2 R_{n-1}^2 (P_{n-1} - P_n)}{R_n^2 - R_{n-1}^2} \frac{1}{r^2}$$

and

$$q = -\frac{P_{n-1} R_{n-1}^2 - P_n R_n^2}{R_n^2 - R_{n-1}^2} + \frac{R_n^2 R_{n-1}^2 (P_{n-1} - P_n)}{R_n^2 - R_{n-1}^2} \frac{1}{r^2},$$

which, being substituted together in the third of the expressions (1), give for the extension of the fibers in the direction of the axis of the cylinder, due to the combined forces acting:

$$\frac{d\Delta h}{dh} = \frac{3PR^2 - 2(P_{n-1} R_{n-1}^2 - P_n R_n^2)}{3(R_n^2 - R_{n-1}^2) E_{n-1}} \qquad (28)$$

Representing the resistance per unit of cross-sectional area by $L$, we may write

$$L = \frac{d\Delta h_{n-1}}{dh_{n-1}} E_{n-1},$$

which will be a true equation of equilibrium so long as the elongation is proportional to the stress. This last, combined with equation (28), gives

$$L = \frac{3PR^2 - 2(P_{n-1}R^2_{n-1} - P_n R^2_n)}{3(R^2_n - R^2_{n-1})} \tag{29}$$

which expresses the longitudinal resistance of the cylinder opposed to the strain in terms of the pressure in the bore of the gun and the interior and exterior normal forces acting directly upon the cylinder.

The limiting value for the longitudinal extension is given by

$$\frac{d\Delta h}{dh} = \frac{\theta_{n-1}}{E_{n-1}}, \text{ or } \theta_{n-1} = \frac{d\Delta h}{dh} E_{n-1}.$$

If, therefore, the value for $L$, derived from equation (29), does not exceed $\theta_{n-1}$, the elastic limit of the metal will not be exceeded. If $L > \theta_{n-1}$, the metal may be permanently set; or if $L$ much exceeds $\theta_{n-1}$, there will be danger of longitudinal rupture.

## Equilibrium of the Gun Structure.

The system composed of a number of layers of simple cylinders superimposed under shrinkage is supposed assembled. Two so-called states of the system have been referred to — the one, that of rest, by which is understood the state which exists after the gun has been assembled and there is no interior pressure acting in the bore; the other, that of action, is the state in which the maximum internal powder-gas pressure acts, this pressure being represented by the greatest value of the internal pressure here to be considered which is consistent with the equilibrium of the structure.

In the first state the only forces acting, except the atmospheric pressure, are the normal pressures at the contact surfaces of the several layers of cylinders composing the gun. In the second state, each of the cylinders is acted upon by an interior and an exterior normal pressure except the outside cylinder, where the atmospheric pressure alone acts upon the outside surface and is considered null. A longitudinal traction also acts in this state of the system, but, as before stated, it will be neglected in considering the equilibrium of the forces acting in directions perpendicular to its line of direction.

We have to discuss the extraneous forces acting upon the elementary cylinders in connection with the displacements of metal which they produce. The equations already deduced expressing the equilibrium of a simple cylinder under the normal pressures $P$ and $P'$ may be applied to any and all of the cylinders in the gun structure, by substituting for $P$ and $P'$ the normal pressures acting upon the surfaces of the cylinder in place in the gun.

Notations: let

$n =$ number of layers of simple cylinders in the system, estimated from the interior outwards.

$P_0 =$ the maximum internal pressure that the system can support.

$P_1, P_2 \dots P_{n-1} =$ the normal pressures at the contact surfaces of the layers during the state of maximum action.

$p_0, p_1 \dots p_{n-1} =$ variations in the pressures $P_0, P_1 \dots P_{n-1}$ from whatever cause produced.

$P'_1, P'_2 \dots P'_{n-1} =$ the normal pressures at contact surfaces during the state of rest.

$p'_1, p'_2 \dots p'_{n-1} =$ variations in the pressures $P'_1, P'_2 \dots P'_{n-1}$ from whatever cause produced.

$R_0, R_1 \dots R_{n-1}, R_n =$ radii of the bounding surfaces of the cylinders.

$\theta_0, \theta_1 \dots \theta_{n-1} =$ elastic limits of metal of cylinder, per unit of area, under tensile stress.*

$\rho_0, \rho_1 \dots \rho_{n-1} =$ elastic limits as above under compression.

$E_0, E_1 \dots E_{n-1} =$ modulii of elasticity of metal of cylinders.

In the treatment of this subject of the construction of a gun of given dimensions and metals it is proposed to take $n = 4$, that is, the case of a gun composed of four layers of cylinders in which the inner cylinder will be designated as the tube, and the remaining ones as the second, third and fourth cylinders respectively. Afterwards the cases of three and of two layers may be considered.

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*The values of $\theta_0$, etc., $\rho_0$, etc., and $E_0 \dots E_{n-1}$ are derived from what may be called *free* tests of the metal; that is, from tests of small specimen bars.

## GUN COMPOSED OF FOUR LAYERS OF CYLINDERS.

The condition to be fulfilled in maintaining the equilibrium of the active and resistant forces in the system is, that the equilibrium of each of the simple cylinders in the structure shall be retained for any state of the system whatever. But in order that each of the cylinders shall contribute its proper share in the resistance of the gun, depending upon the qualities of its metal, we must establish such a relation between the pressures that, as far as practicable, the full capacity of each of the simple cylinders shall be utilized.

The mutual dependence between the normal pressures, considered alone, which the simple cylinders composing the structure would support in the state of maximum action is, for $n = 4$, expressed by the following formulas, derived from (15) and (16):

$$\left.\begin{aligned}
P_3^{(1)} &= \frac{3(R_4^2 - R_3^2)\,\theta_3}{4R_4^2 + R_3^2}, \\
P_2^{(1)} &= \frac{3(R_3^2 - R_2^2)\,\theta_2 + 6R_3^2 P_3}{4R_3^2 + 2R_2^2}, \text{ or } P_2^{(2)} = \frac{3(R_3^2 - R_2^2)\,\rho_2 + 2R_3^2 P_3}{4R_3^2 - 2R_2^2} \\
P_1^{(1)} &= \frac{3(R_2^2 - R_1^2)\,\theta_1 + 6R_2^2 P_2}{4R_2^2 + 2R_1^2}, \text{ or } P_1^{(2)} = \frac{3(R_2^2 - R_1^2)\,\rho_1 + 2R_2^2 P_2}{4R_2^2 - 2R_1^2} \\
P_0^{(1)} &= \frac{3(R_1^2 - R_0^2)\,\theta_0 + 6R_1^2 P_1}{4R_1^2 + 2R_0^2}, \text{ or } P_0^{(2)} = \frac{3(R_1^2 - R_0^2)\,\rho_0 + 2R_1^2 P_1}{4R_1^2 - 2R_0^2}
\end{aligned}\right\} (30)$$

From these the least value in each case must be taken and substituted in the pair of equations following. These least values represent the limiting pressures that the respective cylinders can support; in the one case showing that the metal at the interior surface would reach its limit of elastic displacement by circumferential extension, and in the other by radial compression. The value for $P_0$ derived in this way, however, does not necessarily represent the maximum resistance of the structure, since these equations express no condition governing the safety of the system in the state of rest.

If the system be supposed in action, and we assume the internal pressure $P_0$ removed, the system will pass to the state of rest, and if $p_0$, $p_1$, $p_2$ and $p_3$ represent the variations which occur in the pressures for this transition, the pressures existing in the state of rest will evidently be represented by the algebraic sums:

$$P_0 + p_0,\ P_1 + p_1,\ P_2 + p_2 \text{ and } P_3 + p_3.$$

Suppose the system in a state of equilibrium under action and then subjected to the change indicated; the radii of the elementary cylinders at the several surfaces of contact will undergo certain changes of dimensions, and these changes for any two surfaces in contact will be equal quantities, since the two surfaces are one in effect. These variations in the radii will depend upon the variations in the pressures, and making the proper substitutions in equations (21), (9) and (11), we obtain the following equations expressive of the changes of dimensions:

For the outer surface of the tube cylinder, equation (21) —

$$\frac{\Delta R_1}{R_1} = \frac{6R_0^2 p_0 - (4R_0^2 + 2R_1^2)p_1}{3(R_1^2 - R_0^2)F_0}.$$

For the inner surface of the second cylinder, equation (9) —

$$\frac{\Delta R_1}{R_1} = \frac{(4R_2^2 + 2R_1^2)p_1 - 6R_2^2 p_2}{3(R_2^2 - R_1^2)E_1}.$$

At the contact surface between the second and third cylinders there will be two values of similar form for $\frac{\Delta R_2}{R_2}$ and two also for $\frac{\Delta R_3}{R_3}$ at the outer contact surface, and by equating these separate values we obtain the following equations, of which there is one for each contact surface:

$$\left.\begin{aligned}
\frac{6R_0^2 p_0 - (4R_0^2 + 2R_1^2)p_1}{(R_1^2 - R_0^2)E_0} &= \frac{(4R_2^2 + 2R_1^2)p_1 - 6R_2^2 p_2}{(R_2^2 - R_1^2)E_1} \\
\frac{6R_1^2 p_1 - (4R_1^2 + 2R_2^2)p_2}{(R_2^2 - R_1^2)E_1} &= \frac{(4R_3^2 + 2R_2^2)p_2 - 6R_3^2 p_3}{(R_3^2 - R_2^2)E_2} \\
\frac{6R_2^2 p_2 - (4R_2^2 + 2R_3^2)p_3}{(R_3^2 - R_2^2)E_2} &= \frac{(4R_4^2 + 2R_3^2)p_3}{(R_4^2 - R_3^2)E_3}
\end{aligned}\right\} \quad (32)$$

These equations express the relations between the variations which occur in the pressures at the contact surfaces for a change in the state of equilibrium of the system. By solving them with reference to the values of the variations, beginning with the last, they may be placed under the following forms:

$$\left.\begin{aligned}
p_3 &= \frac{6R_2^2E_3(R_4^2-R_3^2)}{E_3(R_4^2-R_3^2)(4R_2^2+2R_3^2)+E_2(R_3^2-R_2^2)(4R_4^2+2R_3^2)}p_2 \\
p_2 &= \frac{6R_1^2E_2(R_3^2-R_2^2)}{E_2(R_3^2-R_2^2)(4R_1^2+2R_2^2)+E_1(R_2^2-R_1^2)[(4R_3^2+2R_2^2)-6R_3^2\times s]}p_1 \\
p_1 &= \frac{6R_0^2E_1(R_2^2-R_1)}{E_1(R_2^2-R_1^2)(4R_0^2+2R_1^2)+E_0(R_1^2-R_0^2)[(4R_2^2+2R_1^2)-6R_2^2\times t]}p_0
\end{aligned}\right\} (33)$$

in which $s$ represents the value of the fractional coefficient of $p_2$ in the value of $p_3$, and $t$ that of $p_1$ in the value of $p_2$, $s$ and $t$ being each expressed in known terms. There are three of these equations involving four unknown quantities, therefore in order to solve them a given value of one of the unknown quantities, or else a value derived from some other equation of condition, must be introduced. This necessary condition is given by placing

$$p_0 = -P_0$$

in accordance with the present supposition concerning the passage of the system from the state of action to that of rest, and this substitution being made in the last of equations (33), we will be enabled to find the several values for the variations $p_1$, $p_2$ and $p_3$.

It remains now to determine a value for $P_0$ which will be consistent with the safety of the system in both of its extreme states of action and of rest. There is an essential condition affecting the tube cylinder for the state of rest, which is, that the exterior pressure which it has to support in this state of the system shall not be such as will produce an over compression of the bore. The exterior pressure on the tube in this state of the system is represented by $P_1 + p_1$, and its limiting value is given by equations (19) or (20), but usually by the former, which now becomes:

$$P_1 + p_1 = \frac{R_1^2 - R_0^2}{2R_1^2}\rho_0 \qquad (34)$$

Again, the values of the pressures $P_0$ and $P_1$ which the tube cylinder can support with the system in action must be such as will satisfy that one of the last pair of equations (30) which has given the least value for $P_0$. If $\theta_0 =$ or $> \rho_0$, the second of the pair embodying this latter quantity will always be used. We will thus possess one equation containing the two unknown quantities $P_0$ and $P_1$ with known terms. To obtain the second equation necessary to their determination substitute for $p_1$ in equation (34) its value derived from the last of equations (33), having placed in the latter $p_0 = -P_0$. By combining the two equations thus obtained we will determine a value for $P_0$ embodying the conditions required for the two extreme states of the system. From the same equations the corresponding value for $P_1$ will also be derived. The original values of $P_2$ and $P_3$ may or may not be readjusted according to circumstances. They cannot, however, be increased, since they represent the maximum values possible for these pressures. The effect of using the original values of $P_2$ and $P_3$ in connection with reduced values for $P_0$ and $P_1$ will be to increase the relative values of the outer shrinkages and to produce increased strains in the state of rest of the system. If this be deemed undesirable, reduced values for $P_2$ and $P_3$ may be found by assuming reduced values for $\theta_3$ and $\theta_2$ (or $\rho_2$) and applying anew equations (30). The values thus found for $P_2$ and $P_3$ will cause the new working limits of the hoops to be simultaneously developed when the system is in action.

Placing $p_0 = -P_0$ in equations (33), the values of the variations $p_1$, $p_2$ and $p_3$ may be found, and finally the values of the pressures at the contact surfaces for the system at rest. These values are expressed as follows:

$$P_1' = P_1 + p_1, \quad P_2' = P_2 + p_2, \quad P_3' = P_3 + p_3.$$

The value for $P_0$ deduced by the combination indicated above will represent the approximate value of the maximum internal resistance of the system as deduced from these formulas which neglect the longitudinal traction which exists in actual firing. Having obtained the pressures contingent to the state of action, we might proceed to find the shrinkages by the methods given by Clavarino. It is preferred, however, to make use of the pressures incident to the state of rest for reasons which will be stated. The values $P_1'$, $P_2'$ and $P_3'$, representing the pressures which should exist at the contact surfaces of the system in its state of rest, may be substituted in the formulas, and results will be obtained which will be subject to no modification, since these formulas are rigorously applicable to this state of the system wherein the longitudinal traction is both theoretically and practically null. We may then expect to find values for the changes of dimensions incident to the assemblage of the system which will accord exactly with theory. The intimate and direct connection between these pressures and the shrinkages becomes apparent when we consider that the latter are simply the algebraic sums of the changes in radial dimensions produced in adjoining cylinders at each contact surface under the supposition that the pressures $P_1'$, $P_2'$ and $P_3'$ are acting in the two directions upon those surfaces. The shrinkages of a gun thus depend directly upon the values of these pressures.

## PRACTICAL LIMIT FOR THE COMPRESSION OF THE BORE.

We have up to this time assumed that the exterior pressure upon the tube cylinder in the state of rest should be limited by the value derived from formula (34), in which the value used for $\rho_0$ is derived from free tests of the metal. But if it can be shown by practical methods that the compression of the bore, using a given metal for the tube cylinder, can be carried with safety beyond the elastic limit determined by the free tests, then the value of $P_1 + p_1$ or $P_1'$ can be increased accordingly, giving a corresponding increase for the shrinkages and for $P_0$ as well. We may anticipate here by stating that the resistance of a gun composed of given metals will be a maximum for maximum values of the shrinkages, that the greater the shrinkages consistent with the qualities of the metal the greater will be the value of $P_0$ representing the maximum resistance of the structure.

To obtain a measure of this increased value for $P_1'$ it must still be assumed that the compression of the metal is directly proportional to the stress. The direct method will be to find the practical maximum change of dimension that the bore of the tube will safely stand by measuring its compressed state during the experiment. This will give a new value to be used for $\frac{\Delta R_0}{R_0}$; then using equation (12), the value of $P_1 + p_1$ or $P_1'$, its equivalent can be obtained. Assume that having measured the original diameter of the bore and its diameter in the compressed state, a relative compression represented by $\lambda$, which may be considered safe in practice, has been determined. Then from equation (12):

$$\frac{\Delta R_0}{R_0} = \lambda = -\frac{2R_1^2(P_1 + p_1)}{(R'^2 - R_0^2)E_0}$$

whence*

$$P_1' = P_1 + p_1 = \frac{(R_1^2 - R_0^2)\lambda E_0}{2R_1^2} \tag{35}$$

in which $\lambda \times E_0$ takes the place of $\rho_0$ in equation (34).

This equation will then take the place of (34) in the combination used to determine the new values of $P_0$ and $P_1$. Thence we will proceed as before to find the corresponding values of $P_2$ and $P_3$, and, finally, of $P_2'$ and $P_3'$. So long as the internal resistance of the system is limited by the physical qualities of the tube metal, the value of $P_0$ to be found as just indicated will represent the maximum resistance to be obtained from the structure. The corresponding values of $P_1'$, $P_2'$ and $P_3'$ will then be used to deduce the shrinkages consistent with such maximum resistance. Ordinarily the elastic limit of the metal as derived from the free tests will be used, and in that case the shrinkages will be computed from corresponding values for $P_1'$, $P_2'$ and $P_3'$ deduced, as previously explained. Having finally fixed upon the values of the pressures which should exist for the system at rest, all things considered, the next step is to deduce such values for the shrinkages as will produce those pressures, or, otherwise expressed, to find the shrinkages corresponding to those pressures.

## THE SHRINKAGES.

The term shrinkage is used to express the *excess* of the exterior diameter of an inner cylinder over the interior diameter of the cylinder to be placed upon it. Considered as a relative quantity, and expressed in terms of the changes per unit of length which occur at the contact surfaces of the two cylinders when they are assembled, it depends equally upon changes of radii, and this dimension of the circle will be used for the discussion instead of the diameter.

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*The negative sign which should precede $P_1'$ in this equation is omitted. It simply indicates that the pressure thus deduced acts inwards.

Continuing the case of a gun in which $n = 4$; let $\varphi_1$, $\varphi_2$ and $\varphi_3$ represent the successive relative values of the shrinkages counting from the interior. The stages of the construction will be three in number, viz.:

*First stage:* The second cylinder is assembled upon the tube.

*Second stage:* The third cylinder is assembled upon the compound cylinder composed of the two inner cylinders.

*Third stage:* The fourth cylinder is assembled upon the compound cylinder composed of the three inner cylinders.

In each case the interior diameter of the cylinder to be assembled outside is finished bored to the diameter prescribed for the corresponding contact surface, and afterwards the exterior diameter of the surface upon which it is to be assembled is turned to the excess prescribed for the shrinkage. The assemblage is generally made by heating each cylinder to be placed until its interior is sufficiently expanded to enable it to pass over the surface upon which it is to be assembled.

The Pressures acting are

$$P_1', \ P_2' \text{ and } P_3',$$

whose values have been referred to. The forces thus represented will be treated as positive in discussing the values of the shrinkages and radial changes incident to the several stages of the construction.

To express the values of the shrinkages, we have,

$$\left.\begin{array}{l} \varphi_1 = u + u' \, * \\ \varphi_2 = v + v' \\ \varphi_3 = w + w' \end{array}\right\} \qquad (36)$$

In which $u'$, $v'$ and $w'$ represent the relative extension undergone by the interiors of the 2d, 3d and 4th cylinders, respectively, due to the combined action of the interior pressures $P_1'$, $P_2'$, $P_3'$, and the exterior pressures $P_2'$ and $P_3'$; and $u$, $v$, $w$ the relative compressions respectively of the exterior of the tube, the 2d cylinder "assembled and prepared for shrinkage," and the 3d cylinder "assembled and prepared for shrinkage" due to the same pressures

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*This is the algebraic form of the expression. The quantity $u'$ stands for an extension which in this particular case is negative, or a contraction. Hence the numerical form is:

$$\varphi_1 = u - u'$$

$P_1'$, $P_2'$ and $P_3'$. But otherwise expressed, these quantities $u$, $v$, $w$ represent simply the values of the expansions which would take place at the exteriors of these cylinders if we supposed the cylinders exterior to each contact surface (considered separately) removed, for then evidently these exteriors would return to their "original" positions.

The values of $u'$, $v'$, $w'$ can be derived at once from equations (9) and (11) by substituting therein for $P$ and $P'$ the normal pressures to which the several cylinders are now supposed subjected; we thus obtain

$$\left.\begin{aligned} u' &= \frac{\Delta R_1}{R_1} = \frac{(4R_2^2 + 2R_1^2)\,P_1' - 6R_2^2 P_2'}{3(R_2^2 + R_1^2)\,E_1} \\ v' &= \frac{\Delta R_2}{R_2} = \frac{(4R_3^2 + 2R_2^2)\,P_2' - 6R_3^2 P_3'}{3(R_3^2 - R_2^2)\,E_2} \\ w' &= \frac{\Delta R_3}{R_3} + \frac{(4R_4^2 + 2R_3^2)\,P_3'}{3(R_4^2 - R_3^2)\,E_3} \end{aligned}\right\} \qquad (37)$$

To obtain $u$ we must find what would be the expansion of the exterior of the simple tube cylinder if the three outer cylinders were removed. It is equivalent, if we consider the pressure $P_1'$, which is induced by the assemblage of these three cylinders, removed.

The compressing force which acts upon the exterior of the tube is $P_1'$, and the relative compression which it would cause is given, equation (23), by:

$$\frac{\Delta R_1}{R_1} = -\frac{(4R_0^2 + 2R_1^2)\,P_1'}{3(R_1^2 - R_0^2)\,E_0}\,.$$

The compressing force will be rendered null, that is, its removal will be indicated by introducing the condition

$$P_1' = -P_1'$$

which, substituted in the expression above, gives the value of the expansion or restoration sought:

$$u = \frac{(4R_0^2 + 2R_1^2)\,P_1'}{3(R_1^2 - R_0^2)\,E_0} \qquad (38)$$

The pressure supposed acting at the contact surface corresponding to $\varphi_2$ is represented by $P_2'$, which we must suppose removed in order to find the value of $v$. Within this contact surface is a compound cylinder composed of the already assembled tube and 2d cylinder, and the expansion of its exterior due to the removal of $P_2'$ is the quantity sought.

Whilst $P_2'$ is considered acting, the pressure at the contact surface within this compound cylinder is represented by $P_1'$. With these pressures acting on the 2d cylinder the relative change of its exterior, whether an extension or a contraction, is, equation (21), given by:

$$\frac{\Delta R_2}{R_2} = \frac{6R_1^2 P_1' - (4R_1^2 + 2R_2^2) P_2'}{3(R_2^2 - R_1^2) E_1}$$

Supposing the exterior pressure removed from this compound cylinder, which is indicated by substituting

$$P_2' = -P_2',$$

there would occur a ***dependent*** variation in the pressure $P_1'$ at the contact surface within this cylinder, which variation must be found and substituted in the above equation in order to preserve it at the same time with the variation of the exterior pressure indicated by substituting $-p_2'$ for $P_2'$.

The equation of equilibrium for this compound cylinder is given by the first of general equations (32). To apply it in the present case, the following substitutions must be made:

$$p_0 = 0, \; p_1 = p_1', \; p_2 = -P_2';$$

whence is derived the following value for $p_1'$, the variation of $P_1'$, in terms of $P_2'$:

$$p_1' = -\frac{6R_2^2 E_0 (R_1^2 - R_0^2) P_2'}{E_1(R_2^2 - R_1^2)(4R_0^2 + 2R_1^2) + E_0(R_1^2 - R_0^2)(4R_2^2 + 2R_1^2)} \qquad (39)$$

which being substituted for $P_1'$ in the value of $\frac{\Delta R_2}{R_2}$ above at the same time with $P_2' = -P_2'$, we obtain:

$$v' = \frac{(4R_1^2 + 2R_2^2) P_2' - 6R_1^2 p_1'}{3(R_2^2 - R_1^2) E_1} \qquad (40)$$

The pressure supposed acting at the contact surface corresponding to $\varphi_3$ is represented by $P_3'$, which must be supposed removed in order to find the expansion represented by *w*. Within this contact surface is a compound cylinder composed of the already assembled three inner cylinders, and the expansion of its exterior surface due to the